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Convergence rates for diffusive shallow water equations (DSW) using higher order polynomials

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Abstract

In this paper, we describe the diffusive shallow water equation (DSW) and discuss a numerical strategy to solve it using the generalized-α method as a method for temporal discretization. This method provides a good norm estimate of the error and guarantees an optimal convergence rate for the spatial discretization. We also discuss the effect of higher polynomial orders on the convergence rates, focusing on the nonlinear DSW problem. Our numerical experiments show that optional convergence rates can be obtained for polynomial orders 1 through 4.

Keywords: shallow water, barenblatt solutions, convergence rates, DSW

1 Introduction

The shallow water equations are derived from the Navier-Stokes equations by assuming that the vertical momentum scales are small relative to the horizontal ones. Further details can be found in [Vreugdenhil 1994]. The shallow water equations describe and model the propagation of disturbances in water and other incompressible fluids in response to gravitational and rotational accelerations. Their solutions give rise to a system of hyperbolic/parabolic partial differential equations that govern fluid flow in the ocean, coastal regions, estuaries, floods, dam breaks, open channels, rivers, and porous media. They can also be used to predict tides, storm surge levels and coastline changes from hurricanes, ocean currents, atmospheric flow, and to study dredging feasibility. The diffusive shallow water equations (DSW) are derived from the shallow water equations, by assuming that the horizontal momentum can be linked to the water height. Manning's equation is generally used for this purpose [Chow et al 1988]. The DSW is then reduced to a nonlinear diffusion problem of the water height which substitutes the momentum balance equations. This presents challenges but is nonetheless less computationally expensive than solving the shallow water equations [Nochetto and Verdi 1988].

In this paper, we present a solution strategy for the DSW equation using the generalized-α method for temporal evolution of the finite element discretization for horizontal terrain with no source or sink terms. The effect of polynomial order on the spatial discretization is dis- cussed. Our numerical experiments show that optimal convergence rates are obtained for p = 1, 2, 3, 4.

2 Diffusive shallow water equations

2.1 Introduction

The Navier-Stokes equations describe the conservation of mass and momentum for incompressible fluid [Vreugdenhil 1994]. For a free water surface, disregarding the lateral stresses, assuming the simplest possible expression for the bottom stress, and by considering vertical and horizontal scaling, the Navier-Stokes equations will simplify to the shallow water equations which can be written as follows:

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2)+\frac{\partial}{\partial y}(\rho u\upsilon)-(\rho f_1\upsilon)+\frac{\partial P}{\partial x}-\frac{\partial \tau_{xx}}{\partial x}-\frac{\partial \tau_{xy}}{\partial y}=0$$

$$\frac{\partial}{\partial t}(\rho\upsilon)+\frac{\partial}{\partial x}(\rho u\upsilon)+\frac{\partial}{\partial y}(\rho\upsilon^2)-(\rho f_1 u)+\frac{\partial P}{\partial y}-\frac{\partial \tau_{xy}}{\partial x}-\frac{\partial \tau_{yy}}{\partial y}=0$$

where (*x, y*) correspond to the coordinate system shown in Fig.1, t is time , ρ is density, g is the acceleration of gravity, $f_1 = 2\Omega sin\phi$ is the Coriolis parameter which indicates the effect of the rotation of the earth (Ω is the angular rate of revolution, ϕ the geographic latitude), and the viscous stresses τ_{ij} are expressed in terms of the fluid deformation rate as follows.

$$\frac{\tau_{ij}}{\rho}=\nu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \tag{1}$$

The conservation of mass for a fluid element leads to the following mass-conservation equation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho\upsilon)=f(x,y,t) \tag{2}$$

where *f*(*x, y, t*) is the forcing function. The shallow water equations can in turn be simplified

to the diffusive shallow water equation by assuming that [Alonso et al 2008, Feng and Molz 1997, Giammarco et al 1996, Fiedler and Ramirez 2000, Hamrick 1997]:

1. The horizontal momentum is simplified through Manning's equation.
2. The slope of the bathymetry is small.
3. The hydraulic radius can be approximated by the water depth (rectangular cross section).
4. The bottom slope is comparable to the free water surface slope (uniform flow).

The strong form of the DSW model as an initial/boundary-value problem (IBVP) on the domain Ω for times $t \in [0, T]$ is [Alonso et al 2008, Santillana and Dawson 2010]:

$$\begin{aligned}
&\frac{\partial H}{\partial t} - \nabla \cdot \left(\frac{(H-z)^{\alpha}}{C_f \left|\nabla H\right|^{1-\gamma}} \nabla H \right) = f && on && \Omega \times (0,T) \\
&H = H_0 && on && \Omega \times \{t = 0\} \\
&\left(\frac{(H-z)^{\alpha}}{\left|\nabla H\right|^{1-\gamma}} \nabla H \right) \cdot n = B_N && on && \partial\Omega \cap \Gamma_N \times (0,T] \\
&H = B_D && on && \partial\Omega \cap \Gamma_D \times (0,T]
\end{aligned} \quad (3)$$

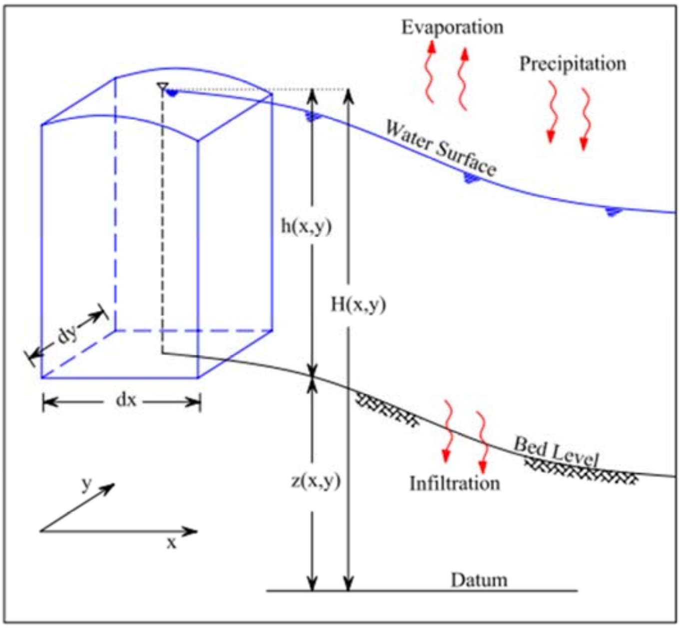

Fig. 1. Control volume for flow through a prism of wetland.

where H is the water height, z is the land surface elevation, f is the forcing function (precipitation acting as a source or infiltration and evaporation acting as sink) as shown in Fig. 1, H_0 is the initial water height distribution, B_N and B_D are the Neumann and Dirichlet conditions, respectively, α and γ are empirical parameters present in Manning's equation [Turner and Chanmeesri 1984], set here to be $\frac{5}{3}$ and $\frac{1}{2}$, respectively and C_f is the Manning roughness coefficient. We assume $C_f = 1$ to compare our results with a self-similar solution proposed by Barenblatt [Grundy and McLaughlin 1784].

Eq. (3) is characterized as doubly nonlinear because inside the divergence term there is a product of two non-linearities consisting of $(H - z)$ and ∇H. For this reason, the previous strong form is rewritten in different alternative forms [Alonso et al 2008, Santillana and Dawson 2010, Esteban and Vazquez 1988].

$$\frac{\partial H}{\partial t} - \nabla \cdot (\alpha(H, \nabla H)\nabla H)) = f \qquad with \qquad \alpha(H, \nabla H) = \frac{(H-z)^{\alpha}}{C_f \left|\nabla H\right|^{1-\gamma}} \tag{4}$$

where α is the diffusion coefficient. The nonlinear character of Eq. (4) is analyzed in [Alonso et al 2008, Santillana and Dawson 2010 a, Santillana and Dawson 2010 b, Calo et al 2011]. New challenges come from the possible degeneracy of a when$(H - z) = 0$, and its nonlinear dependence with respect to ∇H [13]. The DSW equation has been studied in several publications where topographic effects are neglected ($z = 0$) and zero-Dirichlet boundary conditions are assumed [Alonso et al 2008, Esteban and Vazquez 1988].

2.2 Compactly supported solution

Barenblatt made one of the first mathematical contributions to study the initial-value problem for the DSW equation. He constructed a class of self-similar solutions which have the property of finite propagation in 1D, for $z = 0$ and$f = 0$. These solutions will be extensively used later on as a paradigm for the behavior of all non-negative solutions with integrable initial data H_0, especially when H_0 has compact support. More details can be found in [Grundy and McLaughlin 1784, Esteban and Vazquez 1988]. Barenblatt's solution can be expressed as follows:

$$H(x,t) = t^{\left(-\frac{1}{\gamma(m+1)}\right)} \left[C - k(m,\gamma) \,|\, \phi \,|^{\left(\frac{\gamma+1}{\gamma}\right)} \right]_+^{\left(\frac{\gamma}{m(\gamma)-1}\right)} \tag{5}$$

where $[s(x)]_+$ denotes the positive part of $s(x)$, $m = 1 + \frac{\alpha}{\gamma}$, C is a positive function (in this case taken to be equal to 1 [Santillana and Dawson 2010]) which depends on the initial mass M, given by:

$$M = \int_{-\infty}^{\infty} H(x,t)dx$$

$$k(m,\gamma) = \frac{m\gamma}{m(\gamma+1)} \left(\frac{1}{\gamma(m+1)} \right)^{\frac{1}{\gamma}}$$

$$\phi = xt^{-\frac{1}{\gamma(m+1)}}$$

3. Discrete approach

3.1 Space discretization

In the weak form of the DSW, the problem is to find $H \in V$ such that $\forall_w \in W$,

$$R(w,H) = \left(w, \frac{\partial H}{\partial t} \right)_{\Omega} + (\nabla w, \alpha(H, \nabla H)\nabla H)_{\Omega} - (w, f)_{\Omega} = 0.0 \tag{6}$$

where $(.,.)_{\Omega}$ refers to the L^2 inner product. The trial and weighting spaces V and W, respectively, are appropriately chosen for Eq.(6) to be well defned, that is, bounded [Santillana and Dawson 2010]. A discrete approximation to the solution is obtained using Galerkin's method by choosing proper sub-spaces V_h and W_h of V and W, respectively [Hughes 2000]. In this particular case, we chose piecewise polynomial spaces formed by B-splines which have a high degree of continuity at element interfaces.

3.2 Time discretization

When solving highly nonlinear problems, high frequency numerical dissipation has been found to improve the solution behavior [Hughes 2000]. The generalized-α method is a family of time integration algorithms that possess high frequency damping, which can be controlled by the user. It achieves high frequency dissipation while minimizing unwanted low frequency dissipation [Chung and Hulbert 1993]. Due to the non-linearity of the DSW equation and the fact that it is a first order equation in time, we use the temporal discretization method described in [Chung and Hulbert 1993, Jansen et al 2000], which has been used successfully in other complex applications [Gomez et al 2008, Gomez et al2010, Bazilevs et al 2007].

The generalized-α method for problems which are first order in time is stated as [Jansen et al 2000, Gomez et al 2008]: given$(H_n, \dot{H}_n$, find$(H_{n+1}, \dot{H}_{n+1}, H_{n+\alpha_f}, \dot{H}_{n+\alpha_m})$, such that

$$\mathrm{R}(\mathrm{H}_{n+\alpha_f}, \dot{H}_{n+\alpha_m}) = 0 \tag{7}$$

$$\mathrm{H}_{n+\alpha_f} = H_n + \alpha_f(H_{n+1} - H_n) \tag{8}$$

$$\dot{H}_{n+\alpha_m} = \dot{H}_n + \alpha_m(\dot{H}_{n+1} - \dot{H}_n) \tag{9}$$

$$\mathrm{H}_{n+1} = H_n + \nabla t((1-\tilde{\gamma})\dot{H}_n + \tilde{\gamma}\dot{H}_{n+1}) \tag{10}$$

where R is the residual function, $\nabla t = t_{n+1} - t_n$, α_f, α_m, and $\bar{\gamma}$ are parameters of the method. It is useful to combine the two parameters α_m and α_f and express them in terms of the spectral radius for an infinite time step. Chung and Hulbert (1993) referred to this parameter as ρ_∞. Unconditional stability is attained when $\alpha_m \geq \alpha_f \geq \frac{1}{2}$, while second order accuracy is ensured by imposing $\tilde{\gamma} = \frac{1}{2} + \alpha_m - \alpha_f$ as in [17]. The time step size selected follows

$$dt = 0.025\sqrt{dx^{(P+1)}}$$

where *p* is the polynomial order and *dx* is the space step size. The algorithm that describes the generalized-α method is illustrated in Alg.(1), where the consistent tangent matrix K corresponds to the derivative of the residual R with respect to the solution variables [Jansen et al 2000, Gomez et al 2008] and ε is selected to be 10^{-15} . All computations were performed with extended precision 80 bit floating point using PETSc libraries [Balay et al 2010, Balay et al 2008]. The algorithm implementation uses a predictor/multi-corrector scheme where the corrector steps are indicated by a superscript index inside parenthesis. The predictor used in step 1 is one of many possible choices. A time adaptive scheme using this discretization is presented in [Collier et al 2011].

Algorithm 1 Generalized-$\boldsymbol{\alpha}$ method

1: Compute predictor $H_{n+1}^{(0)} = H_n$ and $\dot{H}_{n+1}^{(0)} = \frac{\tilde{\gamma}-1}{\tilde{\gamma}}\dot{H}_n$

2: $H_{n+\alpha_f}^{(0)} = H_n + \alpha_f\left(H_{n+1}^{(0)} - H_n\right)$

3: $\dot{H}_{n+\alpha_m}^{(0)} = \dot{H}_n + \alpha_m\left(\dot{H}_{n+1}^{(0)} - \dot{H}_n\right)$

i=0

while i<maximum iterations **do**

$\quad R^{(i)} = R\left(H^{(i)}_{n+\alpha_f}, \dot{H}^{(i)}_{n+\alpha_m}\right)$

$\quad K^{(i)}_{n+\alpha_f} = \left(\frac{\alpha_m}{\tilde{\gamma}\nabla t\alpha_f}\right)\frac{\partial R(H^{(i)}_{n+\alpha_f}, \dot{H}^{(i)}_{n+\alpha_m})}{\partial \dot{H}_{n+\alpha_m}} + \frac{\partial R(H^{(i)}_{n+\alpha_f}, \dot{H}^{(i)}_{n+\alpha_m})}{\partial H_{n+\alpha_f}}$

$\quad$ Solve $K^{(i)}_{n+\alpha_f}\nabla H^{(i)}_{n+\alpha_f} = -R^{(i)}$

$\quad$ Corrector $H^{(i+1)}_{n+\alpha_f} = H^{(i)}_{n+\alpha_f} + \Delta H^{(i)}_{n+\alpha_f}$

$\quad$ Corrector $\dot{H}^{(i+1)}_{n+\alpha_m} = \left(1 - \frac{\alpha_m}{\tilde{\gamma}}\right)\dot{H}_n + \left(\frac{\alpha_m}{\tilde{\gamma}\,\nabla t\alpha_f}\right)\left(H^{(i+1)}_{n+\alpha_f} - H_n\right)$

$\quad$ **if** $\left\|R^{(i)}\right\| \le \epsilon_\alpha \|R^{(0)}\|$ **then**

$\qquad$ stop

$\quad$ **end if**

$\quad i=i+1$

end while

Solution $H_{n+1} = H_n + \left(\frac{H^{(imax)}_{n+\alpha_f} - H_n}{\alpha_f}\right)$ and $\dot{H}_{n+1} = \dot{H}_n + \left(\frac{\dot{H}^{(imax)}_{n+\alpha_m} - \dot{H}_n}{\alpha_m}\right)$

	$\|\|H_e - H\|\|_{L^2(\Omega)}$				Conver.rate			
dx	p=1	p=2	p=3	p=4	p=1	p=2	p=3	p=4
1	6.06×10^{-3}	4.65×10^{-4}	9.97×10^{-6}	3.60×10^{-6}				
1/2	1.32×10^{-3}	3.23×10^{-5}	3.85×10^{-7}	1.29×10^{-7}	2.19	3.85	4.69	4.80
1/4	2.21×10^{-4}	5.63×10^{-6}	3.70×10^{-8}	4.23×10^{-9}	2.58	2.52	3.38	4.93
1/8	6.75×10^{-5}	6.39×10^{-7}	1.77×10^{-9}	1.36×10^{-10}	1.71	3.14	4.39	4.96
1/16	1.60×10^{-5}	6.83×10^{-8}	1.13×10^{-10}	4.27×10^{-12}	2.08	3.23	3.96	4.99
1/32	3.72×10^{-6}	8.58×10^{-9}	7.17×10^{-12}	1.35×10^{-13}	2.10	2.99	3.98	4.98

Table 1: Norm estimate of errors and convergence rates for higher polynomial orders using the generalized-α method (Non-degenerate case)

4 Convergence rates for higher polynomial orders

Convergence rates of the numerical method proposed to approximate the DSW equation may fail if the depth H - z is zero or if its gradient ∇H is unbounded. Santillana et al.[Santillana and Dawson 2010] proved that one should use at least fourth-order polynomial basis functions in order to ensure their boundedness and thus the convergence of the numerical scheme. This point is checked by using higher polynomial orders p, where p = 1, 2, 3, 4. Table 1 shows the summary of the norm estimates of the error, and the corresponding convergence rates for higher polynomial orders for the non-degenerate case. H_e is the exact solution obtained from Eq. (5). The previous table shows that the convergence rates are of order $p + 1$, and these are plotted in Fig. (2). These simulation results show that optimal rates of convergence can be obtained for polynomial orders lower than four. Thus, the necessary conditions used in [Esteban and Vazquez 1988] may be relaxed to derive a more nuanced approximation theory.

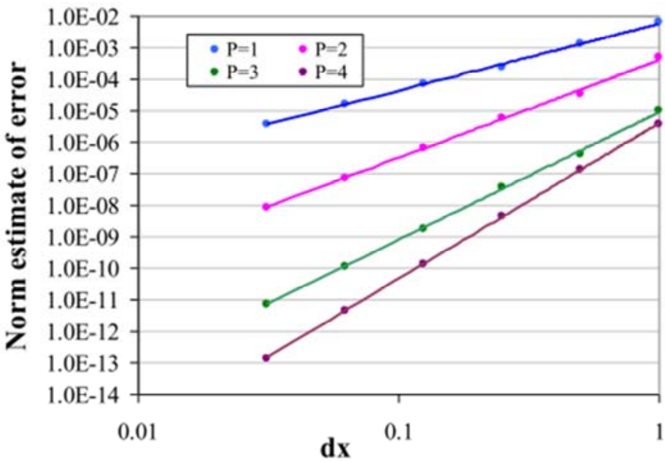

Fig. 2. Convergence rates for higher polynomial orders using the generalized-α method

5 Conclusions

We briefly described the derivation of the diffusive shallow water equations and presented the numerical strategy used for their approximation using the Galerkin finite element procedure for spatial discretization. We then discussed the generalized-α method to be used for temporal discretization. This work shows that the convergence rates for higher order polynomials are of order $p + 1$.

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Извод

Брзине конвергенције за дифузионе једначине за плитку воду (DSW) коришћењем полинома вишег реда

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Резиме

У овом раду дајемо опис дифузионе једначине за плитку воду (DSW) и дискутујемо нумеричку стратегију да решимо ову једначину користећи генералисани $\boldsymbol{\alpha}$ −метод као метод временске дискретизације. Овај метод даје добру процену норме грешке и гарантује оптималну брзину конвергенције за временску дискретизацију. Такође дискутујемо ефекат полинома вишег реда на брзине конвергенције, са посебном пажњом на нелинеарни DSW проблем. Наши нумерички експерименти показују да се оптималне брзине конвергенције могу добити за ред полинома од 1 до 4.

Кључне речи: плитка вода, Баренблатова решења, брзине конвергенције, DSW

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