Euro. Jnl of Applied Mathematics (2008), *vol.* 19, *pp.* 575–606. © 2008 Cambridge University Press
doi:10.1017/S0956792508007675 Printed in the United Kingdom

On the diffusive wave approximation of the shallow water equations†

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(*Received 11 December 2007; revised 20 June 2008; first published online 24 July 2008*)

In this paper, we study basic properties of the diffusive wave approximation of the shallow water equations (DSW). This equation is a doubly non-linear diffusion equation arising in shallow water flow models. It has been used as a model to simulate water flow driven mainly by gravitational forces and dominated by shear stress, that is, under uniform and fully developed turbulent flow conditions. The aim of this work is to present a survey of relevant results coming from the studies of doubly non-linear diffusion equations that can be applied to the DSW equation when topographic effects are *ignored*. In fact, we present proofs of the most relevant results existing in the literature using constructive techniques that directly lead to the implementation of numerical algorithms to obtain approximate solutions.

1 Introduction

In this paper we study some properties of the diffusive wave approximation of the shallow water equations (DSW). This equation arises in shallow water theory when particular flow conditions are assumed. The DSW equation has been used as a model to simulate two-dimensional water flow driven mainly by gravitational forces and dominated by shear stress ([7], [12], [15], [18] and [33]), that is, under uniform and fully developed turbulent flow conditions. These flow conditions occur for example in marshes, wetlands and overland flow in vegetated areas. The DSW equation gives rise to the following initial/boundary-value problem prescribed for any fixed $T > 0$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla \cdot \left(\dfrac{(u-z)^{\alpha}}{|\nabla u|^{1-\gamma}} \nabla u \right) = f & \text{on } \Omega \times (0,T] \\ u = u_0 & \text{on } \Omega \times \{t=0\} \\ \left(\dfrac{(u-z)^{\alpha}}{|\nabla u|^{1-\gamma}} \nabla u \right) \cdot n = g_N & \text{on } \partial\Omega \cap \Gamma_N \times (0,T] \\ u = g_D & \text{on } \partial\Omega \cap \Gamma_D \times (0,T] \end{cases} \tag{1.1}$$

† This work was supported in part by the National Science Foundation, Project No. DMS-0411413, DMS-0620697 and DMS-0636586, and Centro de Investigación en Geografía y Geomática,“Ing. Jorge L. Tamayo”, A.C.

where Ω is an open, bounded subset of $\mathbb{R}^n$ ($n = 1, 2$) and Γ_N and Γ_D are subsets of $\partial\Omega \in C^1$ such that $\partial\Omega = \Gamma_N + \Gamma_D$. Also $f : \Omega \times (0, T] \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $g_N : \Gamma_N \times (0, T] \to \mathbb{R}$, $g_D : \Gamma_D \times (0, T] \to \mathbb{R}$ are given functions, $z : \overline{\Omega} \to \mathbb{R}^+$ is a positive time-independent function, $0 < \gamma \leqslant 1$, $1 < \alpha < 2$ and $u : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is unknown. Here $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ refers to the Euclidean norm in $\mathbb{R}^n$.

Problem (1.1) is characterized as doubly non-linear since the non-linear behaviour appears inside the divergence term as a product of two non-linearities involving $u - z$ and ∇u, namely $(u - z)^\alpha$ and $\nabla u / |\nabla u|^{1-\gamma}$. In fact, when the DSW equation is rewritten as

$$\frac{\partial u}{\partial t} - \nabla \cdot (a(u, \nabla u)\, \nabla u) = f \qquad \text{with} \qquad a(u, \nabla u) = \frac{(u - z)^\alpha}{|\nabla u|^{1-\gamma}},$$

where a is the diffusion coefficient, we can be more specific and characterize it as a doubly non-linear and degenerate-singular equation for the given choice of $0 < \gamma \leqslant 1$ and $1 < \alpha < 2$. This is the case since $a \to 0$ when $(u - z) \to 0$ and $a \to \infty$ when $\nabla u \to 0$.

In the context of shallow water modelling, $u(x, t)$ represents the surface water elevation in the position x at time t, the positive time-independent function $z(x)$ describes the bathymetry of the bed surface throughout the domain and introduces the commonly called *topographic effects* into the model. The topographic effects will be ignored in the mathematical analysis carried out in this paper, i.e. we will assume $z \equiv 0$ in our proofs. Furthermore, to the best of our knowledge, problem (1.1) in its general form (with z being an arbitrary time-independent function) remains an open problem in the PDE literature. See section 5.

In order for equation (1.1) to serve as a suitable model to simulate water flow, two requirements are needed, the first one being that the water depth be non-negative, $u - z \geqslant 0$, and the second one being that the gradient of the water elevation, ∇u, be comparable to the gradient of the bathymetry ∇z, which is usually small. The latter requirement characterizes water flow regimes not far from uniform flow conditions in open channels. The types of physical boundary conditions appropriate for this model are two, a prescribed water depth g_D on Γ_D, and/or a prescribed water flux g_N on Γ_N. The first one corresponds to a Dirichlet-type boundary condition and it is mostly used to model an infinite source of water on the boundary Γ_D. The second one corresponds to a Newmann-type boundary condition and it is the most natural choice to model water flux through a boundary Γ_N.

The outline of the paper is as follows. We begin by providing a brief derivation of problem (1.1) and discussing its relevance in the context of shallow water flow modelling.

We then proceed to present the most relevant results in studies of doubly non-linear diffusion equations existing in the literature that can be applied to the DSW equation when topographic effects are ignored. We present a simple and constructive proof of existence of solutions to the zero-Dirichlet initial/boundary-value problem (1.8) using the Faedo–Galerkin method. This constructive method provides a natural setting for a computational method to find approximate solutions to problem (1.8), further described in [26], within the framework of finite element techniques using piecewise polynomial basis functions. In our proof of existence, instead of following the time-discretization approach established in [17] and [24], we take advantage of the continuous-in-time evolution of the appropriate Banach space norms of the approximate solutions and find *a priori* estimates for them. This is a standard technique proposed in [21] that does not require any truncation–penalization technique as the one used in [6]. Our approximate solutions are *solutions fortes* in the sense of Bamberger [2], and thus, they and their limit will satisfy all the results presented in [2]. In particular, the result on uniqueness of *limite de solutions fortes* in [2] will ensure that the numerical scheme analysed in [26] will converge to a unique solution. See sections 1.2.5 and 4. It is important to note that in our study we do not require the non-linearity in time to be locally Lipschitz as in [6]. In addition, we include a concise argument to prove the L^∞ control and integrability properties of the time derivative of solutions. Although these results have been studied, the regularity arguments we present are hard to find in the literature and provide insight on the complexities of the equation.

For completeness, we include the proof of a comparison result mentioned in [2], and use it to prove uniqueness, non-negativity and stability of the proposed approximation scheme. These findings are then related to problem (1.9) through corollaries and observations. In the last section, we present possible avenues of research as well as conclusions of our study.

1.1 Motivation

Models for surface water flows are derived from the incompressible, three-dimensional Navier–Stokes (NS) equations, which consist of momentum equations for the three velocity components and a continuity equation. Depending on the physics of the flow, scaling arguments are used in order to obtain effective equations for the problem at hand. Equation (1.1) is a simplified version of the two-dimensional shallow water equations called the diffusive wave or zero-inertia approach. This equation is commonly derived by neglecting the inertial terms in the horizontal momentum equations and substituting the bottom slope in Manning's formula by the water surface slope. This approach is shown in [7], [12], [15], [18] and [33]. In the following paragraphs we provide an intuitive, concise and equally valid derivation following a more empirical approach, such as the one used to derive the porous medium equation in section 2 of [28].

Recall that in shallow water theory, the main scaling assumption is that the vertical scales are small relative to the horizontal ones. This approximation reduces the vertical momentum equation to the hydrostatic pressure relation

$$\frac{\partial p}{\partial y} = \rho g, \tag{1.2}$$

where g is the gravitational constant, y the vertical coordinate and p the pressure, and leaves us with two effective momentum equations in the horizontal direction. Upon vertical integration of the NS equations, we obtain two depth-averaged momentum equations and a depth-averaged continuity equation. These resulting equations are called the two-dimensional shallow water equations. For a detailed description of shallow water hydrodynamics see [29] and [31]. When combining the depth-averaged continuity equation with the free surface boundary condition, we obtain the mass balance equation

$$\frac{\partial h}{\partial t} + \nabla \cdot (hV) = f, \tag{1.3}$$

where $h(x,t) = H(x,t) - z(x)$ is the water depth, $H(x,t)$ is the free water surface elevation or hydraulic head, $z(x)$ is the bed surface, bathymetry or land elevation, $\boldsymbol{V}(x,t)$ is the depth-averaged velocity and $f(x,t)$ is a source/sink (such as rainfall or infiltration).

In open channel flow theory, empirical laws such as Manning's formula or Chézy's formula have been observed to successfully describe the dynamics of water flow in regimes when fluid motion is dominated by gravity and balanced by the bottom boundary shear stress. See [20] or chapter 11 in [10]. Examples of open channel flow include water flow in rivers, in partially full drains and surface runoff. Manning's and Chézy's formulas relate the mean velocity of the flow V with the so-called *hydraulic radius*[1] R and the *bottom slope* S through a friction coefficient c_f in the following way:

$$V = \frac{1}{c_f} R^{\alpha-1} S^{\gamma}, \tag{1.4}$$

for particular choices of α and γ. For Manning's formula[2] $\alpha = 5/3$ and $\gamma = 1/2$, and for Chézy's formula $\alpha = 3/2$ and $\gamma = 1/2$. When we multiply equation (1.4) by the hydraulic radius R, we obtain an equivalent relation in terms of the water discharge Q

$$Q = RV = \frac{1}{c_f} R^{\alpha} S^{\gamma}. \tag{1.5}$$

The discharge-depth equation (1.5) is a generalization of both Manning's formula or Chézy's formulas and was proposed in [27] as a way to account for more general circumstances when flow changes back and forth between turbulent and laminar conditions. This is the case, for example, in water flow in vegetated areas. In [27], Turner and Chanmeesri study equation (1.5) as a prediction model for shallow water flow in vegetated areas based entirely on empirical procedures. In their study they conclude that equation (1.5) with flexible coefficients α and γ results in a broader and better model than the particular Manning's formula. Experimentally, they reported values in the ranges $1 \leqslant \alpha \leqslant 2$ and $0 < \gamma < 1$. These values motivate the ranges of α and γ in the present work. Further assumptions in open channel theory that justify the application of velocity-depth equations like (1.4) include:

- the approximation of the hydraulic radius R by the water depth h in (1.4) and (1.5),
- the assumption that the slope of the bathymetry is small and
- the assumption that the bottom slope is comparable to the free water surface slope.

In the diffusive wave approximation, we make use of the previous assumptions, and extend the scaling of the mean flow velocity V with respect to R and S in (1.4), to the depth-averaged velocity $\boldsymbol{V}(x,t)$ in (1.3) along the direction of the flow. This is done in the following way: since the flow is assumed to be dominated by gravity, the direction of the flow will be along the unitary vector $\nabla H/|\nabla H|$ [recall equation (1.2)], and thus, equation (1.4)

[1] The hydraulic radius for open channels is calculated as $R = A/\omega$, where A is the cross section of the channel, and ω is the wetted perimeter. Note that for a rectangular cross section with base L and depth h, $R = hL/(L+2h) \sim h$ when $L \gg h$.

[2] For a derivation of Manning's formula based on the phenomenological theory of turbulence see [16].

is transformed into

$$V = -\frac{h^{\alpha-1}}{c_f}\frac{\nabla H}{|\nabla H|}|\nabla H|^{\gamma} = -\frac{(H-z)^{\alpha-1}}{c_f}\frac{\nabla H}{|\nabla H|^{1-\gamma}}, \tag{1.6}$$

The DSW equation is given by a doubly non-linear parabolic equation for the water elevation H, obtained from substituting the particular form of the depth-averaged horizontal velocity given by (1.6), into equation (1.3)

$$\frac{\partial H}{\partial t} - \nabla\cdot\left(\frac{(H-z)^{\alpha}}{c_f}\frac{\nabla H}{|\nabla H|^{1-\gamma}}\right) = f(t,x), \qquad \text{for } (t,x)\in\mathbb{R}^+\times\mathbb{R}^2, \tag{1.7}$$

The assumptions made to obtain the DSW equation suggest that it may be suitable to serve as a model in low-to-moderate velocity-flow regimes. See section 2 of [12] and the references therein.

Remark 1.1 Note that if one identifies the water elevation H with the hydrostatic pressure p, the expression that relates the velocity and the water elevation gradient (1.6) becomes a non-linear version of the empirical *Darcy's law* for gas flow through a porous medium. Indeed, flow in vegetated areas such as wetlands can be understood as a flow through a porous medium.

Remark 1.2 Note in particular that for the case when $\gamma = 1$, $c_f \equiv 1$ and $z \equiv 0$, equation (1.7) becomes the porous medium equation (PME). One should expect similarities between the PME and the more general equation (1.7), although some differences may arise. See section 1.2. A comprehensive study of the PME can be found in the book by Vázquez [28]. Another relevant particular case of equation (1.7) arises when $\alpha = 0$ and $c_f \equiv 1$. This equation is commonly known as the time evolution equation of the p-Laplacian [here $(\gamma+1 = p)$]. An important reference addressing this equation can be found in DiBenedetto [11].

1.2 Prior work

To the best of our knowledge, the DSW equation has not been studied in its general form (1.1). However, when topographic effects are neglected ($z \equiv 0$) and zero-Dirichlet initial/boundary conditions are assumed ($\partial\Omega = \Gamma_D$), we can find a fairly extensive number of works that study doubly non-linear equations that are relevant to the DSW equation. See for example [2], [17], [19], [21] and [24]. Most of these works study alternative formulations of problem (1.1). These will be explained in the subsequent sections. In this paper we will focus our attention on the alternative formulation given by

$$\begin{cases} \dfrac{\partial\phi(v)}{\partial t} - \eta^{\gamma}\,\nabla\cdot\left(\dfrac{\nabla v}{|\nabla v|^{1-\gamma}}\right) = f & \text{on } \Omega\times(0,T] \\ v = 0 & \text{on } \partial\Omega\times[0,T] \\ v = v_0 & \text{on } \Omega\times\{t=0\} \end{cases} \tag{1.8}$$

where Ω is either $\mathbb{R}^n$ or an open (and in most cases bounded) subset of $\mathbb{R}^n$, η is a positive constant and the function $\phi(s) \in C^{0,\eta}(\mathbb{R})$ is an *odd* function satisfying the following properties:

(i) $|\phi(s)| \leqslant |s|^\eta$ for $0 < \eta \leqslant \gamma < 1$, with equality for $|s| \geqslant R$ for some $R \geqslant 0$.

(ii) $\phi(s)$ is a concave increasing function for $s \geqslant 0$.

Note that with the change of variables defined by $u = \phi(v)$, problem (1.8) is transformed into

$$\begin{cases} \dfrac{\partial u}{\partial t} - \eta^\gamma \, \nabla \cdot \left(((\phi^{-1})'(u))^\gamma \dfrac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f & \text{on } \Omega \times (0, T] \\ u = 0 & \text{on } \partial\Omega \times [0, T] \, . \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \end{cases} \tag{1.9}$$

Now, choosing

$$0 < \eta = \frac{\gamma}{\alpha + \gamma} < 1 \qquad \text{and} \qquad \phi(s) = \frac{s}{|s|^{1-\eta}} \tag{1.10}$$

we can obtain the explicit expression for

$$(\phi^{-1})'(s) = (1 + \theta)|s|^{\,\theta} \qquad \text{where} \qquad \theta = \frac{1 - \eta}{\eta} = \frac{\alpha}{\gamma} \tag{1.11}$$

which yields the following equation:

$$\frac{\partial u}{\partial t} - \nabla \cdot \left(|u|^\alpha \frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f. \tag{1.12}$$

The previous manipulations lead us to conclude that, at least formally, non-negative solutions of problem (1.8) are solutions of the original problem (1.1) under the aforementioned assumptions.

At this point it is important to clarify the scope of the present work regarding its relevance within the shallow water modelling context. A complete analysis of the DSW equation and thus problem (1.1), should be posed as an obstacle problem, in other words, any physical solution u of problem (1.1) should be greater than or equal to the topography z (in this paper considered flat) regardless of the sign of the input f (possibly negative when modelling physical processes such as *infiltration* or *evaporation*). Note that a solution u of problem (1.8) [recall equation (1.12)] could be negative, and thus physically inconsistent. However, as we will show in section 4, the non-negativity of f will imply the non-negativity of a solution u of problem (1.8), for any physically consistent initial condition $u_0 \geqslant 0$. Furthermore, our analysis will be relevant even in cases when the combination of inputs (*infiltration, evaporation* and *rainfall*) are such that $u \geqslant 0$. A classical example of an obstacle problem approach can be found in [28] for the PME, where free boundary issues need to be explicitly addressed. A closely related one-dimensional obstacle problem formulation can be found in [8] for a doubly non-linear parabolic equation arising in ice sheet dynamics. In general, the theory of free boundaries is an important and difficult subject of mathematical investigation. In particular, the free boundary theory for doubly non-linear equations is an area of research far from being complete.

1.2.1 *Existence of solutions*

Lions [21] introduced the techniques of compactness and monotonicity later utilized in the subsequent works in the proofs of existence for problem (1.8). Raviart [24] and Grange and Mignot [17] prove the existence of weak solutions to problem (1.8), provided Ω is an open and bounded subset of $\mathbb{R}^n$, constructing approximate solutions using implicit finite differences schemes in time and passing to the limit by means of compactness and monotonicity. In [24], Raviart worked directly with problem (1.8), and in [17], Grange and Mignot extended such results to the abstract setting of equations of the type

$$\frac{\partial Bu}{\partial t} + Au = f,$$

where A and B denote the subdifferentials of convex functionals. Their analysis is based on the essential restriction that these functionals must be continuous on appropriate Banach spaces. Bernis further extends these results to the case when Ω is any open set of $\mathbb{R}^n$ in [5]. Another relevant reference is [6], where Blanchard and Francfort address the semi-abstract problem

$$\frac{\partial}{\partial t} b(u) - \nabla \cdot (D\Phi(\nabla u)) = f,$$

where b is a locally Lipschitz function and may grow faster than any power function at infinity, and Φ is a C^1 convex functional with specific coercivity assumptions. They obtain existence and comparison results with the aid of a Galerkin approximation technique which uses truncation–penalization of the time non-linearity and *a priori* estimates through convex conjugate functions. An important work addressing quasi-linear and doubly non-linear parabolic equations is found in Alt and Luckhaus [1].

1.2.2 *Comparison principles and uniqueness*

In [2], Bamberger studies the existence of particular solutions to problem (1.8) which are the limit of *solutions fortes* i.e. solutions that have the property $\phi(u)_t \in L^1(0, T, L^1(\Omega))$. Bamberger refers to this kind of solutions as *limite de solutions fortes*. In addition, he presents a very concise exposition of a comparison principle between solutions that are *limite de solutions fortes* and uses this result to find uniqueness. See section 4.

1.2.3 *Regularity*

When topographic effects are neglected ($z \equiv 0$) and zero-Dirichlet initial/boundary conditions are assumed ($\partial\Omega = \Gamma_D$) Problem (1.1) can also be rewritten in the form

$$\frac{\partial u}{\partial t} - \nabla \cdot (|\nabla u^m|^{\gamma-1} \nabla u^m) = f$$

with $m = 1 + \alpha/\gamma$. Esteban and Vázquez [13] studied this equation in one-dimensional for the Cauchy problem ($\Omega = \mathbb{R}$). They study the *local velocity of propagation*

$$V(x,t) = -v_x |v_x|^{\gamma-1},$$

where v is the non-linear potential defined as

$$v = \begin{cases} \dfrac{m\gamma}{m\gamma - 1}\, u^{\frac{m\gamma-1}{\gamma}} & if \quad m\gamma \neq 1 \\ \dfrac{1}{\gamma}\, \log u & if \quad m\gamma = 1 \end{cases}.$$

Recall that in the DSW equation, it is assumed that $m\gamma = (\alpha + \gamma) > 1$. In their work, they base their approach on the existing theory for the PME and find the estimate

$$V_x \leqslant \frac{1}{\gamma(m+1)\,t}.$$

Using the previous estimate as the main tool, they construct a theory for the Cauchy problem with non-negative, integrable initial data. In particular, they address the following questions:

- existence, uniqueness and regularity of strong solutions,
- existence and regularity of free boundaries,
- asymptotic behaviour of solutions and free boundaries.

In [19], Ishige gives a sufficient condition for the growth order of the initial data at infinity for the existence of weak solutions of the Cauchy problem ($\Omega = \mathbb{R}$) (1.8).

1.2.4 *Additional properties of solutions*

Some interesting facts about non-negative solutions to problem (1.8) are

- *Finite speed of propagation.* Indeed, Barenblatt constructed a class of self-similar source-type solutions for the Cauchy problem ($\Omega = \mathbb{R}$) which have the property that their supports propagate in time with finite speed, when $(\alpha + \gamma) > 1$. See [3].
- *Extinction property.* In [2], using simple arguments, Bamberger exhibits that for $f = 0$, non-negative solutions to the zero-Dirichlet boundary-value problem ($\Omega \subset \mathbb{R}$, bounded) become zero in finite time.
- *Travelling waves.* It is worthwhile mentioning that an interesting example of travelling-wave-type solutions

$$u(x,t) = U(t - n \cdot x) \qquad \text{with} \quad U(s) = 0 \qquad \text{for} \quad s > 0,$$

  to the zero-Dirichlet boundary-value problem ($\Omega \subset \mathbb{R}$, bounded) is shown in [2] for the case when $\eta > \gamma$ [equivalently $\alpha < (1 - \gamma)$]. In the DSW equation this case does not arise since $\alpha > 1$ and $0 < \gamma \leqslant 1$.

Other properties of solutions including *non-existence of global non-negative solutions* and *blow up solutions* can be found in [5] and [19], respectively, for particular choices of the parameters η and γ that do not happen in the DSW equation case.

1.2.5 *Numerical methods*

For completeness in our presentation we proceed to mention some studies where similar regularization techniques, such as the one presented in this work, have been used to implement numerical methods to approximate the solutions of equations related to the DSW. In [22] for example, Nochetto and Verdi study a class of degenerate parabolic equations (such as the Stefan problem and the PME) of the form

$$\frac{\partial u}{\partial t} - \nabla \cdot (\nabla v + b(v)) + f(v) = 0, \qquad u \in m(v), \tag{1.13}$$

where $m(v)$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ possibly with a singularity at the origin $[m'(0) = \infty]$. In their numerical analysis they use a smoothing procedure that regularizes m. Other relevant numerical studies using some sort of regularization techniques in the approximation of solutions of degenerate parabolic equations include [4], [23], [25] and [30]. Of particular interest is the work presented in [26], which may be thought of as the numerical analysis counterpart of the current paper. In this work, Santillana and Dawson implement a methodology inspired by the regularization techniques proposed in this paper to numerically approximate the solution of the DSW equation in the context of shallow water modelling. In [26], *a priori* error estimates between the regularized solution of the DSW and fully discrete solutions are obtained under certain regularity and physically consistent conditions. The qualitative behaviour of solutions to problem (1.1), including *topographic effects* is investigated numerically. Furthermore, numerical experiments that provide relevant information about the numerical accuracy of the method and the applicability of the DSW equation as a model to simulate observed quantities in a real-life experimental setting are presented. Rainfall is considered in [26], however, neither evaporation nor infiltration are investigated.

1.3 Notation

We will use the standard notation introduced in [14]. Let X be a real Banach space, with norm $\|\cdot\|$. The symbol $L^p(0,T;X)$ will denote the Banach space of all measurable functions $u : [0,T] \to X$ such that

(i) $\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|^p\right)^{1/p} < \infty, \qquad$ for $1 \leqslant p < \infty$ and
(ii) $\|u\|_{L^\infty(0,T;X)} := \operatorname{ess\,sup}_{0 \leqslant t \leqslant T} \|u(t)\| < \infty.$

We will denote with $C([0,T];X)$ the space of all continuous functions $u : [0,T] \to X$ such that

$$\|u\|_{C(0,T;X)} := \max_{0 \leqslant t \leqslant T} \|u(t)\| < \infty.$$

Let $u \in L^1(0,T;X)$, we say $v \in L^1(0,T;X)$ is the weak time derivative of u, denoted as $u_t = v$, provided

$$\int_0^T \psi_t(t)\, u(t) = -\int_0^T \psi(t)\, v(t)$$

for all scalar test functions $\psi \in C_0^\infty(0,T)$. Throughout the paper, $W^{1,p}(0,T;X)$ will denote the space of all functions $u \in L^p(0,T;X)$ such that u_t exists in the weak sense and $u_t \in L^p(0,T;X)$ with the norm

$$\|u\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|u(t)\|^p + \|u_t(t)\|^p\right)^{1/p} & (1 \leqslant p < \infty), \\ \operatorname*{ess\,sup}_{0\leqslant t\leqslant T} (\|u(t)\| + \|u_t(t)\|) & (p = \infty). \end{cases}$$

For $1 \leqslant p \leqslant +\infty$, we will denote its conjugate as p^* i.e., $1/p + 1/p^* = 1$. For any measurable set $E \subset \Omega$ and real-valued vector functions $u \in L^p(E)$ and $v \in L^{p^*}(E)$ we will denote the duality pairing between u and v as

$$(u,v)_E := \int_E u \cdot v.$$

For simplicity, we use $(u,v) := (u,v)_\Omega$. Similarly, we will denote the duality pairing between $u \in W^{-1,p^*}(\Omega)$ and $v \in W_0^{1,p}(\Omega)$ as $\langle u,v \rangle$. Recall that the elements of $W^{-1,p^*}(\Omega)$ are the distributions that have continuous extention to $W_0^{1,p}(\Omega)$. These spaces are characterized in the following way: if $u \in W^{-1,p^*}(\Omega)$, then there exists functions $f^0, f^1, \ldots, f^n$ in $L^{p^*}(\Omega)$ such that

$$\langle u,v \rangle = (f^0, v) + \sum_{i=1}^n (f^i, v_{x_i}).$$

Throughout the paper, C will be a generic constant with different values, and the explicit dependence with respect to parameters will be written inside parenthesis.

1.4 Definitions of weak solution

From now on, we will assume that $\phi(s)$ and η are given by (1.10), and $0 < \gamma \leqslant 1$, $1 < \alpha < 2$.

Definition 1.1 We say a function

$$v \in L^{1+\gamma}\big(0,T; W_0^{1,(1+\gamma)}(\Omega)\big), \text{ with } \phi(v)_t \in L^{(1+\gamma)^*}(0,T; W^{-1,(1+\gamma)^*}(\Omega)),$$

is a weak solution of the initial/boundary-value problem (1.8) provided

$$\langle \phi(v)_t, w \rangle + \eta^\gamma \left(\frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w \right) = (f, w) \qquad \text{a.e. in time } 0 \leqslant t \leqslant T, \tag{1.14}$$

for any $w \in W_0^{1,(1+\gamma)}(\Omega)$ and

$$v(0) = v_0. \tag{1.15}$$

Definition 1.2 We say a function u, with the properties

$$\phi^{-1}(u) \in L^{1+\gamma}\big(0, T; W_0^{1,1+\gamma}(\Omega)\big), \text{ and } u_t \in L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega)),$$

is a weak solution of the initial/boundary-value problem (1.9) provided

$$\langle u_t, w\rangle + \eta^\gamma \left(((\phi^{-1})'(u))^\gamma \frac{\nabla u}{|\nabla u|^{1-\gamma}}, \nabla w \right) = (f, w) \qquad \text{a.e. in time } 0 \leqslant t \leqslant T, \tag{1.16}$$

for any $w \in W_0^{1,(1+\gamma)}(\Omega)$ and

$$u(0) = u_0. \tag{1.17}$$

Remark 1.3 A consequence of Definition 1.1 [resp. (1.2)] is that

$$\phi(v) \in C([0, T]; W^{-1,(1+\gamma)^*}(\Omega)) \qquad [\text{ resp. } u \in C([0, T]; W^{-1,(1+\gamma)^*}(\Omega))\,]$$

thus condition (1.15) [resp. (1.17)] makes sense.

Remark 1.4 In Definition 1.2, we understand the pointwise gradient of u, denoted as ∇u, as the function

$$\nabla u = \begin{cases} \phi'(v)\nabla v & \text{if } |v| > 0 \\ 0 & \text{if } v = 0 \end{cases}$$

where $v \in L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$ is a weak solution of the initial/boundary-value problem (1.8).

2 Existence

In order to prove the existence of a weak solution of problem (1.8) we will use the Faedo–Galerkin method using compactness and monotonicity arguments as explained in [21]. The method consists of five main steps.

Step 1. Constructing approximate solutions by the method of Faedo–Galerkin.

Step 2. Finding *a priori* estimates on such approximate solutions.

Step 3. Using the properties of compactness to extract a converging subsequence to pass to the limit.

Steps 4 and 5. Using the monotonicity of the non-linear operator $\mathscr{A}(\mathbf{x})$ (see the appendix) to prove that the limit process indeed leads to a weak solution.

The key idea of the proof is to find a solution to problem (1.8) when ϕ is replaced by a Lipschitz function ϕ_{reg} approximating ϕ uniformly, such that $|\phi_{\text{reg}}| \leqslant |\phi|$. Throughout the paper we will refer to any of these approximations as *regular* ϕ or ϕ_{reg}, indistinctively. Then, we will show that a solution to problem (1.8) can be found as a limit of these *regularized* solutions. For the sake of clarity, the reader can think of the following family of regularized Lipschitz functions:

$$\phi_{\text{reg}} = \phi_\epsilon(s) = \begin{cases} \dfrac{\phi(\epsilon)}{\epsilon} s & \text{if } |s| \leqslant \epsilon, \\ \phi(s) & \text{if } |s| > \epsilon. \end{cases}$$

Clearly, $\phi_\epsilon(s) \to \phi(s)$ uniformly as $\epsilon \to 0$. In fact,

$$\sup_s |\phi(s) - \phi_\epsilon(s)| \leqslant \epsilon^\eta.$$

With the previous ideas in mind, Steps 1, 3 and 4 will be performed for any regular ϕ_{reg}, the *a priori* estimates obtained in Step 2 will be computed for ϕ and thus, they will hold uniformly for any ϕ_{reg}. The latter fact will allow us to find in Step 5 a subsequence of *regularized* solutions that will converge to a solution of problem (1.8).

Theorem 2.1 *Let f and v_0 satisfy*

$$v_0 \in L^{1+\eta}(\Omega) \qquad \text{and} \qquad f \in L^{(1+\gamma)^*}(0, T; L^{(1+\gamma)^*}(\Omega)), \tag{2.1}$$

then there exists a function v with the properties

$$v \in L^{(1+\gamma)}\big(0, T; W_0^{1,(1+\gamma)}(\Omega)\big), \tag{2.2}$$

and

$$\phi(v)_t \in L^{(1+\gamma)^*}(0, T; W^{-1,(1+\gamma)^*}(\Omega)), \tag{2.3}$$

such that it solves problem (1.8).

Proof For clarity we organize the proof in the steps previously described.

Step 1: Approximate Solutions

Let $\{w_j\}_{j=1}^{\infty}$ be a basis of $V = W_0^{1,(1+\gamma)}(\Omega)$. Construct the Faedo–Galerkin approximate solution of problem (1.8), $v_m(t)$, the following way. For any fixed t

$$v_m(t) = \sum_{j=1}^{m} \zeta_j(t) w_j(x) \in [w_1, \ldots, w_m] = \text{the space generated by } \{w_j\}_{j=0}^{m}$$

and satisfying

$$(\phi(v_m)_t, w_j) + \eta^\gamma \left(\frac{\nabla v_m}{|\nabla v_m|^{1-\gamma}}, \nabla w_j \right) = (f, w_j) \qquad 1 \leqslant j \leqslant m, \tag{2.4}$$

$$v_m(0) = v_{0,m} \in [w_1, \ldots, w_m],$$

where $v_{0,m} \to v_0$ in $L^{1+\eta}(\Omega)$.

Step 2: *A priori* **Estimates**

Lemma 2.1 *Set $\phi(s) = s/|s|^{1-\eta}$. Let v_m be a Faedo–Galerkin approximate solution of problem* (1.8)*, then the following estimates hold.*

$$\sup_{0\leqslant t\leqslant T} \|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \leqslant C(\|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T) \tag{2.5}$$

and

$$\|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}^{1+\gamma} \leqslant C(\|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T) \tag{2.6}$$

where $(1+\eta)^* = (1+\eta)/\eta$.

Proof Multiply equation (2.4) by $\zeta_j(t)$ and sum for $1 \leqslant j \leqslant m$ to obtain

$$\frac{d}{dt}\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1+\eta}{\eta^{1-\gamma}}\int_\Omega |\nabla v_m|^{1+\gamma} = \frac{1+\eta}{\eta}(f, v_m) \tag{2.7}$$

and from Young's inequality

$$(f, v_m) \leqslant \frac{\eta}{1+\eta}\|f\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1}{1+\eta}\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*}. \tag{2.8}$$

Now, since

$$\int_\Omega |\nabla v_m|^{1+\gamma} \geqslant 0$$

we get the inequality

$$\frac{d}{dt}\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \leqslant \|f\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1}{\eta}\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*}.$$

Using Gronwall's lemma we get that for all $t \in [0, T]$

$$\|\phi(v_m)(t)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \leqslant C(\|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T)$$

which leads to the first estimate stated in (2.5).

Note: We have assumed, without loss of generality, that

$$\|v_{0,m}\|_{L^{1+\eta}(\Omega)} \leqslant \|v_0\|_{L^{1+\eta}(\Omega)}.$$

Integrating equation (2.7) in time

$$\|\phi(v_m)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \frac{1+\eta}{\eta^{1-\gamma}}\int_0^T\int_\Omega |\nabla v_m|^{1+\gamma} = \frac{1+\eta}{\eta}\int_0^T (f, v_m) + \|v_{0,m}\|_{L^{1+\eta}(\Omega)}^{1+\eta}.$$

The above expression and inequality (2.8) imply that

$$\|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))}^{1+\gamma} \leqslant C(\|v_0\|_{L^{1+\eta}(\Omega)}, \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}, T)$$

which finishes the proof. □

Remark 2.1 Note that by the Poincaré inequality

$$\|v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))} \leqslant C(\Omega)\,\|\nabla v_m\|_{L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega))},$$

therefore the sequence $\{v_m\} \subset L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$ and it is uniformly bounded.

Step 3: Passing to the Limit

Let $v_m(t)$ be the Faedo–Galerkin sequence of approximate solutions of problem (1.8) defined by (2.4). Estimates (2.5) and (2.6) in Lemma 2.1 imply that there exists a convergent subsequence $\{v_\mu\}$ of $\{v_m\}$ such that

$$v_\mu \rightharpoonup v \qquad \text{in } L^{1+\gamma}\big(0,T;W_0^{1,1+\gamma}(\Omega)\big) \text{ weakly,} \tag{2.9}$$

$$\phi(v_\mu)(T) \rightharpoonup \xi \qquad \text{in} \qquad L^{(1+\eta)^*}(\Omega) \qquad \text{weakly.} \tag{2.10}$$

In addition, inequality (2.6) implies

$$\frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}} \rightharpoonup \chi \qquad \text{in} \qquad L^{(1+\gamma)^*}(0,T;L^{(1+\gamma)^*}(\Omega)) \text{ weakly.} \tag{2.11}$$

Integrating equation (2.4) in time and using the aforementioned convergence results, we can take the limit $\mu \to \infty$ to find that for any $w \in L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$

$$\lim_{\mu\to\infty} \int_0^T (\phi(v_\mu)_t, w) = -\,\eta^\gamma \int_0^T (\chi,\nabla w) + \int_0^T (f\,,w). \tag{2.12}$$

We can conclude that

$$\phi(v_\mu)_t \rightharpoonup \vartheta \qquad \text{in} \qquad L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)) \qquad \text{weakly,} \tag{2.13}$$

where the functional ϑ is defined by the right-hand side of equation (2.12). Using (2.9) and Theorem A.2, we can conclude that

$$\phi(v)_t = \vartheta. \tag{2.14}$$

Therefore, for any $w \in L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$

$$\int_0^T \langle \phi(v)_t, w\rangle = -\,\eta^\gamma \int_0^T (\chi,\nabla w) + \int_0^T (f\,,w). \tag{2.15}$$

Note also that $L^{(1+\gamma)/\eta}(\Omega) \subset W^{-1,(1+\gamma)^*}(\Omega)$, hence we have

$$\phi(v) \in L^{(1+\gamma)^*}(0,T;L^{(1+\gamma)/\eta}(\Omega)) \subset L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)).$$

Using the previous fact, together with (2.13) and (2.14)

$$\phi(v) \in W^{1,(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)).$$

So by Theorem A.1 we conclude that

$$\phi(v) \in C([0, T]; W^{-1,(1+\gamma)^*}(\Omega))$$

and

$$\phi(v)(t) - \phi(v)(s) = \int_s^t \phi(v)_t \ \text{ for all } \ 0 \leqslant s \leqslant t \leqslant T. \tag{2.16}$$

Multiply equation (2.16) by $w \in W^{1,1+\gamma}(\Omega)$ and integrate in Ω to obtain

$$\begin{aligned}\langle \phi(v)(T) - \phi(v_0), w \rangle &= \int_0^T \langle \phi(v)_t, w \rangle \\ &= \lim_{\mu\to\infty} \int_0^T (\phi(v_\mu)_t, w) \\ &= \lim_{\mu\to\infty} (\phi(v_\mu)(T) - \phi(v_{0,\mu}), w) \\ &= \langle \xi - \phi(v_0), w \rangle .\end{aligned}$$

Since w is arbitrary, we conclude that

$$\phi(v)(T) = \xi. \tag{2.17}$$

Step 4: Monotonicity Argument

It only remains to show that

$$\chi = \frac{\nabla v}{|\nabla v|^{1-\gamma}}$$

in equation (2.15). For that purpose, recall by the monotonicity Lemma A.1 that for any $w \in L^{1+\gamma}(0, T; W_0^{1,1+\gamma}(\Omega))$

$$X_\mu \equiv \eta^\gamma \int_0^T \left(\frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}} - \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v_\mu - \nabla w \right) \geqslant 0$$

which we can rewrite as

$$X_\mu = T_{1,\mu} + T_{2,\mu}$$

where

$$T_{1,\mu} = \eta^\gamma \int_0^T \left(\frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}}, \nabla v_\mu \right)$$

and

$$T_{2,\mu} = -\eta^\gamma \int_0^T \left(\frac{\nabla v_\mu}{|\nabla v_\mu|^{1-\gamma}}, \nabla w \right) - \eta^\gamma \int_0^T \left(\frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v_\mu - \nabla w \right).$$

Note that

$$\limsup_\mu X_\mu = \limsup_\mu T_{1,\mu} + \limsup_\mu T_{2,\mu} \geqslant 0. \tag{2.18}$$

From (2.9) and (2.11) we can easily see that

$$\limsup_\mu T_{2,\mu} = \lim_\mu T_{2,\mu} = -\eta^\gamma \int_0^T (\chi, \nabla w) - \eta^\gamma \int_0^T \left(\frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v - \nabla w \right). \tag{2.19}$$

For the term $T_{1,\mu}$ we need to be more careful. Using equation (2.4)

$$\begin{aligned} T_{1,\mu} &= -\int_0^T (\phi(v_\mu)_t, v_\mu) + \int_0^T (f, v_\mu) \\ &= -\frac{\eta}{\eta+1}\int_0^T \frac{d}{dt}\,\|\phi(v_\mu)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \int_0^T (f, v_\mu) \\ &= \frac{\eta}{\eta+1}\,\|\phi(v_{0,\mu})\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \frac{\eta}{\eta+1}\,\|\phi(v_\mu)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} + \int_0^T (f, v_\mu). \end{aligned}$$

Since by (2.17) and a well-known property of weak limits

$$\|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)} = \|\xi\|_{L^{(1+\eta)^*}(\Omega)} \leqslant \liminf_{\mu} \|\phi(v_\mu)(T)\|_{L^{(1+\eta)^*}(\Omega)}.$$

Thus, we get

$$\limsup_{\mu} T_{1,\mu} \leqslant \frac{\eta}{\eta+1}\left(\|\phi(v_0)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \right) + \int_0^T (f, v).$$

Now, substitute v for w in (2.15). Perform the integration in time to find that

$$\eta^\gamma \int_0^T (\chi, \nabla v) = \frac{\eta}{\eta+1}\left(\|\phi(v_0)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} - \|\phi(v)(T)\|_{L^{(1+\eta)^*}(\Omega)}^{(1+\eta)^*} \right) + \int_0^T (f, v). \tag{2.20}$$

Thus, from (2.18), (2.19) and (2.20) we observe that

$$\int_0^T \left(\chi - \frac{\nabla w}{|\nabla w|^{1-\gamma}}, \nabla v - \nabla w\right) \geqslant 0;$$

if we choose $w = v - \lambda\psi$ for $\lambda > 0$ and $\psi \in L^{1+\gamma}(0,T; W_0^{1,1+\gamma}(\Omega))$ in the previous equation, then

$$\int_0^T \left(\chi - \frac{\nabla(v-\lambda\psi)}{|\nabla(v-\lambda\psi)|^{1-\gamma}}, \nabla\psi\right) \geqslant 0.$$

Taking the limit as $\lambda \to 0$ we finally obtain that

$$\int_0^T \left(\chi - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla\psi\right) \geqslant 0$$

which implies by Lebesgue's lemma that

$$\chi = \frac{\nabla v}{|\nabla v|^{1-\gamma}}.$$

The previous fact completes the proof of Theorem 2.1 for any ϕ_{reg}.

Step 5: Going from ϕ_{reg} to ϕ

Next, take $\{\phi_k\}_{k=1}^{\infty}$ to be a sequence of regularized functions converging uniformly to $\phi(s) = s/|s|^{1-\eta}$. Then, *a priori* estimates (2.5) and (2.6), which are independent of k, hold for the sequences $\{\phi_k(v_k)\}$ and $\{v_k\}$. Hence, Steps 3 and 4 can be identically performed to find that v defined as

$$v = \lim_{k\to\infty} v_k$$

is a weak solution of the problem for the non-regular ϕ. □

Corollary 2.1 *There exists a weak solution to problem* (1.9), *where the gradient of u is understood as the pointwise gradient.*

Proof Let v be a weak solution of problem (1.8) with initial condition $v_0 = \phi^{-1}(u_0)$ and let $u = \phi(v)$. Immediately, the following holds:

(i) $u = 0$ in $(0, T) \times \partial\Omega$,

(ii) $u(0) = \phi(v(0)) = \phi(v_0) = \phi(\phi^{-1}(u_0)) = u_0$,

(iii) $\phi(v)_t = u_t$.

It only remains to show that the weak gradient of v and the pointwise gradient of u are related by

(iv) $\nabla v = (\phi^{-1})'(u)\nabla u \qquad$ a.e. in $(0, T) \times \Omega$.

For this purpose, observe that since $v \in L^{1+\gamma}(0, T; W_0^{1,(1+\gamma)}(\Omega))$ there exists a sequence $v_m \in L^{1+\gamma}(0, T; C^\infty(\Omega))$ such that

$$v_m \to v \quad \text{strongly in} \quad L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)) \quad \text{and a.e. in } (0, T) \times \Omega.$$

Define the sequence $u_m = \phi(v_m)$. Since $v_m \in L^{1+\gamma}(0, T; C^\infty(\Omega))$, the following relation holds true a.e.

$$\nabla u_m = \begin{cases} \phi'(v_m)\nabla v_m & \text{if} \quad |v_m| > 0, \\ 0 & \text{if} \quad v_m = 0. \end{cases}$$

Therefore,

$$u_m \to u \qquad \text{and} \qquad \nabla u_m \to \nabla u \qquad \text{a.e. in } (0, T) \times \Omega.$$

In addition, $v_m = \phi^{-1}(u_m)$, thus

$$\nabla v_m = (\phi^{-1})'(u_m)\, \nabla u_m \qquad \text{a.e. in } (0, T) \times \Omega.$$

Taking $m \to \infty$ in the previous expression, we find that

$$\nabla v = (\phi^{-1})'(u)\, \nabla u \qquad \text{in } L^{1+\gamma}(0, T; L^{1+\gamma}(\Omega)).$$

To conclude the proof, substitute (iii) and (iv) in equation (1.14) to obtain equation (1.16). □

Remark 2.2 As pointed out in equations (1.10) and (1.12) an immediate consequence of Corollary 2.1 is that if u is a non-negative solution of problem (1.8) then it solves problem (1.9) in the sense of Definition 1.2.

Corollary 2.2 *Let v be a weak solution of the initial/boundary-value problem* (1.8). *Then for any $w \in L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$*

$$\langle \phi(v)_t, w\rangle + \left(\frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w\right) = (f,w) \quad \text{a.e. in } [0,T].$$

Proof Fix $w \in L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$ and let $\{w_j\}$ be a basis for $W_0^{1,(1+\gamma)}(\Omega)$. Take a sequence $\{\psi_m\}$ of the form

$$\psi_m = \sum_{j=1}^{m} d_j^m(t) w_j \quad \text{with} \quad d_j^m(t) \in L^\infty([0,T])$$

such that $\psi_m \to w$ strongly in $L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$. This is possible by density of such finite sums in the mentioned space.

Since v is weak solution of problem (1.8) we get

$$\langle \phi(v)_t, \psi_m\rangle + \left(\frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla \psi_m\right) = (f,\psi_m) \quad \text{a.e. in } [0,T].$$

Take $m \to +\infty$ to conclude. □

3 Regularity

In this section, we investigate basic regularity properties of solutions found in the existence Theorem 2.1. It is desirable to find more information on the time derivative of the function $\phi(v)$, in particular, it is worthwhile to find that it is a regular distribution.

Theorem 3.1 *Assume*

$$v_0 \in W_0^{1,1+\gamma}(\Omega), \quad \text{and} \quad f \in L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega)).$$

Let v be a solution of problem (1.8) as constructed in Theorem 2.1, then

(i) $v \in L^\infty(0,T;W_0^{1+\gamma}(\Omega))$,

(ii) v_t exists as a regular distribution that lies in $L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$

with the estimate

$$\int_{\{|v|>0\}} \left(\phi'(v)^{1/2} v_t\right)^2 + \sup_{[0,T]} \|\nabla v(t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leqslant C\left(T,\ \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))},\ \|v_0\|_{W_0^{1,1+\gamma}(\Omega)}\right). \tag{3.1}$$

Moreover, when ϕ is regular then $\phi(v)_t$ also lies in $L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$ and

$$\phi(v)_t = \phi'(v) v_t. \tag{3.2}$$

Proof Let $\{\phi_k\}_{k=1}^{\infty}$ be a sequence of regularized functions converging uniformly to $\phi(s) = s/|s|^{1-\eta}$ and let $v_k(t)$ be the solution associated to each ϕ_k. Then

$$(\phi_k(v_k)_t, (v_k)_t) + \eta^\gamma \left(\frac{\nabla v_k}{|\nabla v_k|^{1-\gamma}}, \nabla (v_k)_t \right) = (f, (v_k)_t).$$

Hence,

$$\left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{\eta^\gamma}{1+\gamma} \frac{d}{dt} \|\nabla v_k\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} = (f, (v_k)_t).$$

In addition, note that

$$(f, (v_k)_t) \leqslant \frac{1}{2} \left\| \frac{f}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2.$$

Thus, combining the last two relations we get

$$\frac{1}{2} \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{\eta^\gamma}{1+\gamma} \frac{d}{dt} \|\nabla v_k\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \leqslant \frac{1}{2} \left\| \frac{f}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2. \tag{3.3}$$

Integrating (3.3) in time from 0 to T, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^T \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{\eta^\gamma}{1+\gamma} \sup_{[0,T]} \|\nabla v_k(t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \\ &\quad \leqslant \frac{1}{2} \int_0^T \left\| \frac{f}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^{1+\gamma}(\Omega)}^{1+\gamma}. \end{aligned} \tag{3.4}$$

By the hypothesis imposed on f, the right-hand side of (3.4) converges to

$$\frac{1}{2} \int_0^T \left\| \frac{f}{\phi'(v)^{1/2}} \right\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \quad \text{as} \quad k \to \infty.$$

This immediately implies that the right-hand side is bounded. Because of the non-linearities that occur in the left-hand side of (3.4), it is not straightforward to send $k \to \infty$ to establish estimate (3.1). For this purpose, we will first establish a weak convergence result for the sequence $\{(v_k)_t\}$ in the following way.

Observe that since $\phi(v_k)_t = \phi_k'(v_k)(v_k)_t$ then

$$\int_0^T \|(v_k)_t\|_{L^{1+\eta}(\Omega)}^{1+\eta} \leqslant \frac{1+\eta}{2} \int_0^T \left\| \frac{\phi_k(v_k)_t}{\phi_k'(v_k)^{1/2}} \right\|_{L^2(\Omega)}^2 + \frac{1-\eta}{2} \int_0^T \|1/\phi_k'(v_k)\|_{L^q(\Omega)}^q \tag{3.5}$$

where $q = (1+\eta)/(1-\eta)$. Note that

$$\phi'(s) = \frac{\eta}{|s|^{1-\eta}}, \tag{3.6}$$

therefore

$$\frac{1}{\phi'(s)} = \frac{|\phi(s)|^{\frac{1-\eta}{\eta}}}{\eta} \quad \text{and} \quad \int_0^T \|1/\phi_k'(v_k)\|_{L^q(\Omega)}^q = \frac{1}{\eta^{\,q}} \|\phi_k(v_k)\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}^{(1+\eta)^*}. \tag{3.7}$$

Hence, as a consequence of (3.4), (3.5) and (3.7) the sequence $\{(v_k)_t\}$ is bounded in $L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$. Thus, there exists a subsequence of $\{(v_k)_t\}$, labelled with the index μ such that

$$(v_\mu)_t \rightharpoonup v_t \ \text{ weakly in } L^{1+\eta}(0,T;L^{1+\eta}(\Omega)) \text{ as } \ \mu \to +\infty. \tag{3.8}$$

Second, define for all $\epsilon > 0$ and $m \geqslant 1$ the set

$$\Omega_{m,\epsilon} := \bigcap_{j\geqslant m}^{+\infty} \{[0,T]\times\Omega : |v_j| \geqslant \epsilon\}.$$

Thus,

$$\begin{aligned}\int_0^T \left\| \frac{\phi_\mu(v_\mu)_t}{\phi_\mu'(v_\mu)^{1/2}} \right\|_{L^2(\Omega)}^2 &= \int_0^T \left\| \phi_\mu'(v_\mu)^{1/2}\,(v_\mu)_t \right\|_{L^2(\Omega)}^2 \\ &\geqslant \int_{\Omega_{m,\epsilon}} \left(\phi_\mu'(v_\mu)^{1/2}\,(v_\mu)_t\right)^2. \end{aligned}\tag{3.9}$$

Now, in $\Omega_{m,\epsilon}$ we have the bound $\phi_\mu'(v_\mu)^{1/2} \leqslant \eta\,\epsilon^{\eta-1}$ for $\mu \geqslant m$ and clearly,

$$\phi_\mu'(v_\mu)^{1/2} \to \phi'(v)^{1/2} \ \text{ a.e. in } \Omega_{m,\epsilon}.$$

Using this fact with (3.8) we obtain

$$\phi_\mu'(v_\mu)^{1/2}\,(v_\mu)_t \rightharpoonup \phi'(v)^{1/2}\,v_t \ \text{ weakly in } L^{1+\eta}(\Omega_{m,\epsilon}).$$

Therefore, taking $\liminf_{\mu\to+\infty}$ in (3.9) and using the weakly lower semi-continuity property of convex functionals on L^p it follows that

$$\int_{\Omega_{m,\epsilon}} \left(\phi'(v)^{1/2}\,v_t\right)^2 \leqslant \liminf_{\mu\to+\infty} \int_0^T \left\| \frac{\phi_\mu(v_\mu)_t}{\phi_\mu'(v_\mu)^{1/2}} \right\|_{L^2(\Omega)}^2. \tag{3.10}$$

As $v_j \to v$ a.e. in $[0,T]\times\Omega$, it follows that

$$\lim_{m\to\infty,\epsilon\to 0} \Omega_{m,\epsilon} = \{|v| > 0\}.$$

Hence, taking these limits in (3.10) we obtain

$$\int_{\{|v|>0\}} \left(\phi'(v)^{1/2} v_t\right)^2 \leqslant \liminf_{\mu\to+\infty} \int_0^T \left\| \frac{\phi_\mu(v_\mu)_t}{\phi_\mu'(v_\mu)^{1/2}} \right\|_{L^2(\Omega)}^2. \tag{3.11}$$

This takes care of the first term in (3.4). The second term of the left-hand side is simpler to deal with. Note that by (3.4) there exist a subsequence of $\{v_k\}$, labelled again with the index μ, such that

$$v_\mu \rightharpoonup \xi \text{ in } L^\infty\big(0,T;W_0^{1,1+\gamma}(\Omega)\big) \text{ weak}^*.$$

Since the sequence already converged weakly in $L^{1+\gamma}(0,T;W_0^{1,1+\gamma}(\Omega))$ to v, we conclude that $\xi = v$. Therefore, we can take $\liminf_{\mu\to+\infty}$ in (3.4) to obtain

$$\begin{aligned}\frac{1}{2}\int_{\{|v|>0\}}\big(\phi'(v)^{1/2}v_t\big)^2 + \frac{\eta^\gamma}{1+\gamma}\sup_{[0,T]}\|\nabla v(t)\|_{L^{1+\gamma}(\Omega)}^{1+\gamma} \\ \leqslant \frac{1}{2}\int_0^T\left\|\frac{f}{\phi'(v)^{1/2}}\right\|_{L^2(\Omega)}^2 + \|\nabla v_0\|_{L^{1+\gamma}(\Omega)}^{1+\gamma}. \end{aligned} \tag{3.12}$$

To get estimate (3.1), observe that using the first expression in (3.7) we can prove, using Hölder's inequality, that

$$\int_0^T\left\|\frac{f}{\phi'(v)^{1/2}}\right\|_{L^2(\Omega)}^2 \leqslant \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}^2\,\|\phi(v)\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))}^{\frac{1-\eta}{\eta}}$$

which together with estimate (2.5) prove (i), (ii) and estimate (3.1). Finally when ϕ is regular, it is Lipschitz, then the chain rule formula in (3.2) follows by a standard result for Sobolev functions. □

Corollary 3.1 *Assume the conditions of Theorem 3.1. Then for any regular ϕ,*

$$\phi(v)_t \in L^2(0,T;L^2(\Omega)),$$

and the following estimate holds

$$\|\phi(v)_t\|_{L^2(0,T;L^2(\Omega))}^2 \leqslant C\big(T,\ \phi'(0),\ \|f\|_{L^{(1+\eta)^*}(0,T;L^{(1+\eta)^*}(\Omega))},\ \|v_0\|_{W_0^{1,1+\gamma}(\Omega)}\big).$$

Proof The conditions on any regular ϕ imply that for any $s \in \mathbb{R}$

$$1 \leqslant \frac{\phi'(0)}{\phi'(s)}$$

thus, after applying the chain rule (3.2) in Theorem 3.1, it follows that

$$\begin{aligned}\int_{\{|v|>0\}}\phi(v)_t^2 &= \int_{\{|v|>0\}}\big(\phi'(v)v_t\big)^2 \\ &\leqslant \phi'(0)\int_{\{|v|>0\}}\big(\phi'(v)^{1/2}v_t\big)^2.\end{aligned}$$

In addition, observe that in the set $\{v=0\}$ we have $\phi(v)=0$. Hence, a direct calculation shows that $\phi(v)_t = 0$ in the interior of this set. But $\phi(v)_t$ is measurable, therefore

$$\int_{\{v=0\}}\phi(v)_t^2 = 0.$$

Consequently,

$$\|\phi(v)_t\|^2_{L^2(0,T;L^2(\Omega))} \leqslant \phi'(0)\int_{\{|v|>0\}} \left(\phi'(v)^{1/2}v_t\right)^2.$$

Using estimate (3.1) in Theorem 3.1 we conclude the proof. □

Remark 3.1 Corollary 3.1 shows that the solutions of problem (1.8) constructed as in Theorem 2.1 are *solutions fortes* in the sense of [2].

Theorem 3.2 *Assume v is a solution of problem (1.8) constructed as in Theorem 2.1, and additionally assume that*

$$v_0 \in L^\infty(\Omega) \qquad and \qquad f \in L^\infty(0,T;L^\infty(\Omega)),$$

then

$$\sup_{t\in[0,T]} \|v(t)\|_{L^\infty(\Omega)} \leqslant C\left(\|v_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T\right). \tag{3.13}$$

Proof In order to find an L^∞ bound on v, we would like to uniformly control its L^p norms. For this purpose, the key idea would be to multiply equation (1.8) by the test function $v/|v|^{1-a}$ for any $a \geqslant 1$ and use Gronwall's Lemma to establish the result. However, for a fixed time t, the test function $v/|v|^{1-a}$ does not necessarily belong to $W_0^{1,1+\gamma}(\Omega)$, so that we need to regularize it. For this end, let us introduce the family $\{\rho_\delta(s)\}_{\delta>0}$ approximating the function $s/|s|^{1-a}$

$$\rho_\delta(s) = \frac{1}{(1+\delta|s|)^a}\,\frac{s}{|s|^{1-a}}.$$

Note that $\rho_\delta(v)(t) \in L^{1+\gamma}(0,T;W_0^{1,(1+\gamma)}(\Omega))$ since $\rho_\delta(s)$ is a $C^1([0,\infty))$ function with bounded derivative.

Using Corollary (2.2) we can chose $\rho_\delta(v)$ as a test function in equation (1.8). Observe that for any regular ϕ, the solution v has time derivative $v_t \in L^{1+\eta}(0,T;L^{1+\eta}(\Omega))$ by Theorem (3.1), hence the chain rules applies,

$$\phi(v)_t = \phi'(v)v_t.$$

Therefore, the following relation holds immediately

$$\frac{d}{dt}\|\Phi_\delta(v)(t)\|_{L^1(\Omega)} = \langle \phi(v)_t, \Phi_\delta(v)\rangle$$

where

$$\Phi_\delta(s) = \int_0^s \phi'(z)\rho_\delta(z).$$

Thus, we obtain

$$\frac{d}{dt}\|\Phi_\delta(v)(t)\|_{L^1(\Omega)} + \eta^\gamma(|\nabla v|^{1+\gamma}, \rho'_\delta(v)) = (f, \rho_\delta(v)). \tag{3.14}$$

The second term in the left-hand side of (3.14) is non-negative, thus the following inequality holds

$$\frac{d}{dt}\|\Phi_\delta(v)(t)\|_{L^1(\Omega)} \leqslant \|f\|_{L^\infty(\Omega)}\|\rho_\delta(v)\|_{L^1(\Omega)}.$$

Using the fact that

$$|\rho_\delta(s)| \leqslant 1 + \frac{\eta + a}{\eta}\Phi_\delta(s),$$

we obtain from the previous relation that

$$\frac{d}{dt}X_\delta(t) \leqslant \|f(t)\|_{L^\infty(\Omega)}\left(|\Omega| + \frac{\eta + a}{\eta}X_\delta(t)\right),$$

where

$$X_\delta(t) = \|\Phi_\delta(v)(t)\|_{L^1(\Omega)}.$$

Using Gronwall's lemma we get

$$X_\delta(t) \leqslant \exp\left(\frac{\eta + a}{\eta}\|f\|_{L^\infty(0,T;L^\infty(\Omega))}T\right)\left\{X_\delta(0) + \|f\|_{L^\infty(0,T;L^\infty(\Omega))}T\right\}. \tag{3.15}$$

Inequality (3.15) is valid for any ϕ_{reg}. Similarly, observe that

$$\Phi_\delta(v)(t) \longrightarrow \frac{\eta}{\eta + a}|v|^{\eta+a}(t) \text{ pointwise as } \delta \to 0 \text{ in } [0, T] \times \Omega.$$

Thus, taking $\delta \to 0$ in (3.15) and using Fatou's Lemma it follows that

$$\frac{\eta}{\eta + a}\|v(t)\|^{\eta+a}_{L^{\eta+a}(\Omega)} \leqslant \exp\left(\frac{\eta + a}{\eta}\|f\|_{L^\infty(0,T;L^\infty(\Omega))}T\right)\left\{\frac{\eta}{\eta + a}\|v_0\|^{\eta+a}_{L^{\eta+a}(\Omega)} + |\Omega|\|f\|_{L^\infty(0,T;L^\infty(\Omega))}T\right\}. \tag{3.16}$$

Taking the $\eta + a$ *root* in (3.16) and letting $a \to \infty$ we find that for $0 \leqslant t \leqslant T$

$$\|v(t)\|_{L^\infty(\Omega)} \leqslant \exp(\eta^{-1}\|f\|_{L^\infty(0,T;L^\infty(\Omega))}\ T)\max(1, \|v_0\|_{L^\infty(\Omega)}) \tag{3.17}$$

which proves the result for any regular ϕ. Next, take $\{\phi_k\}_{k=1}^\infty$ to be a sequence of regularized functions converging uniformly to $\phi(s) = s/|s|^{1-\eta}$. Let v_k be the solution associated to each ϕ_k, then, as in the proof of existence,

$$v = \lim_{k\to\infty} v_k \qquad \text{pointwise in} \quad (0, T) \times \Omega.$$

Thus, estimate (3.17) holds for v. This concludes the proof. □

Corollary 3.2 *Assume u is a solution of problem (1.9) found as in Corollary 2.1, and additionally assume that*

$$u_0 \in L^\infty(\Omega) \qquad \textit{and} \qquad f \in L^\infty(0, T; L^\infty(\Omega)),$$

then

$$\sup_{t\in[0,T]} \|u\|_{L^\infty(\Omega)} \leqslant C(\|u_0\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T).$$

Proof The solution for problem (1.9) found as in Corollary 2.1 is given by $u = \phi(v)$, thus by estimate (3.13) we have

$$\sup_{t\in[0,T]} \|\phi^{-1}(u)\|_{L^\infty(\Omega)} \leqslant C(\|\phi^{-1}(u_0)\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T). \tag{3.18}$$

Since ϕ^{-1} is a monotonically increasing function, we have the property that

$$\sup_{t\in[0,T]} \|\phi^{-1}(u)\|_{L^\infty(\Omega)} = \phi^{-1}\big(\sup_{t\in[0,T]} \|u\|_{L^\infty(\Omega)}\big).$$

Substituting the previous fact and applying ϕ on both sides of (3.18) we obtain

$$\begin{aligned}\sup_{t\in[0,T]} \|u\|_{L^\infty(\Omega)} &\leqslant \phi(C(\phi^{-1}(\|(u_0)\|_{L^\infty(\Omega)}), \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T)),\\ &\leqslant C(\|(u_0)\|_{L^\infty(\Omega)}, \|f\|_{L^\infty(0,T;L^\infty(\Omega))}, T)\end{aligned}$$

which finishes the proof. □

4 Comparison result, uniqueness and non-negativity

Generally speaking, if v is a weak solution of problem (1.8) some basic regularity on $\phi(v)_t$ must be obtained for pursuing a uniqueness result, otherwise this task can be very complex. Moreover, uniqueness may not be true. In Theorem (4.1) we will prove a comparison result due to Bamberger [2] that will lead to a uniqueness result under the assumption that

$$\phi(u)_t \in L^1(0,T;L^1(\Omega)). \tag{4.1}$$

In a hydrologic context, the previous assumption can be interpreted in the following way. Condition (4.1) implies that $u_t \in L^1(0,T;L^1(\Omega))$ in problem (1.9). Hence

$$u \in C(0,T;L^1(\Omega)) \subseteq W^{1,1}(0,T;L^1(\Omega)).$$

Recall that when u is non-negative, u represents the free water surface elevation, or the column of water at a given point in the domain Ω in a physical system. Thus the volume $\mathscr{V}$ of water in Ω may be represented as

$$\mathscr{V}(\Omega,t) = \int_\Omega u(t).$$

Condition (4.1) implies that the the volume in the domain Ω changes continuously in time. This is a natural condition when modelling hydrologic systems. The fact that the volume is a time-continuous function follows when integrating expression (ii) of Theorem A.1 to obtain

$$\mathscr{V}(t_1) - \mathscr{V}(t_0) = \int_{t_0}^{t_1} \mathscr{V}_t(t),$$

where

$$\mathscr{V}_t(\Omega, t) = \int_\Omega u_t(t) \quad \in L^1(0, T).$$

Therefore $\mathscr{V}$ is an absolutely continuous function in $[0, T]$.

In the current section, we will use the standard notation f^+ and f^- to denote the positive and negative part of the function f, respectively.

Theorem 4.1 *(Ref. [2]). Assume u and v are weak solutions of problem (1.8) associated to the initial data u_0 and v_0, and the forcing terms f and g, respectively. Assume the additional property that*

$$\phi(u)_t,\ \phi(v)_t \in L^1(0, T; L^1(\Omega)) \quad \textit{and} \quad f - g \in L^1(0, T; L^1(\Omega)), \tag{4.2}$$

then

$$\int_\Omega \lambda(\phi(u) - \phi(v)) \leqslant \int_\Omega \lambda(\phi(u_0) - \phi(v_0)) + \int_0^t \int_\Omega \lambda(f - g), \tag{4.3}$$

where $\lambda(s)$ is any of the following three functions, $|s|$, s^+ or s^-.

Proof Since u and v are weak solutions of problem (1.8) then

$$\langle \phi(u)_t - \phi(v)_t, w\rangle + \left(\frac{\nabla u}{|\nabla u|^{1-\gamma}} - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \nabla w\right) = (f - g, w) \tag{4.4}$$

for any $w \in W_0^{1,(1+\gamma)}(\Omega))$. Let $\{\beta_\delta(s)\}_{\delta>0}$ be the family of $C^1(\mathbb{R})$ increasing functions such that,

(i) $|\beta_\delta(s)| \leqslant 1$, and
(ii) $\beta_\delta(s) \longrightarrow \lambda'(s)$ as $\delta \to \infty$.

Substituting $w = \beta_\delta(u - v)$ in (4.4) we find that

$$\langle \phi(u)_t - \phi(v)_t, \beta_\delta(u - v)\rangle + \left(\frac{\nabla u}{|\nabla u|^{1-\gamma}} - \frac{\nabla v}{|\nabla v|^{1-\gamma}}, \beta'_\delta(u - v)\nabla(u - v)\right) = \big(f - g, \beta_\delta(u - v)\big).$$

Since $\beta'_\delta(u - v) \geqslant 0$, by Lemma A.1, the second term in the previous expression is non-negative, thus

$$\int_0^t \langle \phi(u)_t - \phi(v)_t, \beta_\delta(u - v)\rangle \leqslant \int_0^t (f - g, \beta_\delta(u - v)).$$

Note that $\{\beta_\delta(u - v)\} \subset L^\infty(0, T; L^\infty(\Omega))$. But $\phi(u)_t$ and $\phi(v)_t$ lie in $L^\infty(0, T; L^\infty(\Omega))^*$ by assumption, thus

$$\langle \phi(u)_t - \phi(v)_t, \beta_\delta(u - v)\rangle = (\phi(u)_t - \phi(v)_t, \beta_\delta(u - v)).$$

Using Lebesgue's dominated convergence theorem we can take the limit as $\delta \to \infty$ in the above inequality to find that

$$\int_0^t (\phi(u)_t - \phi(v)_t, \lambda'(u-v)) \leqslant \int_0^t \int_\Omega \lambda(f-g).$$

Observe that since $\lambda'(u-v) = \lambda'(\phi(u) - \phi(v))$, then for $0 \leqslant t \leqslant T$,

$$\begin{aligned}\int_0^t (\phi(u)_t - \phi(v)_t, \lambda'(u-v)) &= \int_0^t ((\phi(u) - \phi(v))_t, \lambda'(\phi(u) - \phi(v)))\\ &= \int_0^t \frac{d}{dt} \int_\Omega \lambda(\phi(u) - \phi(v))\\ &= \int_\Omega \lambda(\phi(u)(t) - \phi(v)(t)) - \int_\Omega \lambda(\phi(u_0) - \phi(v_0)), \qquad (4.5)\end{aligned}$$

from which (4.3) follows. □

Remark 4.1 By hypothesis $\phi(v) \in C([0, T]; L^1(\Omega))$ since $\phi(v) \in W^{1,1}([0, T]; L^1(\Omega))$. See Theorem A.1. Thus, the last step in (4.5) can be safely performed.

Remark 4.2 Note that if ϕ is regular we know from Corollary 3.1 that solutions of problem (1.8) constructed as in Theorem 2.1 satisfy

$$\phi(u)_t,\ \phi(v)_t \in L^2(0, T; L^2(\Omega)) \subset L^1(0, T; L^1(\Omega)).$$

Hence, the previous result applies for them.

Corollary 4.1 (Uniqueness) *Assume u and v are weak solutions of problem (1.8) satisfying*

$$\phi(u)_t,\ \phi(v)_t \in L^1(0, T; L^1(\Omega)),$$

then $u = v$.

Proof Use Theorem 4.1 with $\lambda(s) = |s|$, $u_0 = v_0$ and $f = g$. □

Corollary 4.2 *Assume u and v are weak solutions of problem (1.8) associated to the initial data* u_0 *and* v_0*, and the forcing terms f and g, respectively. Also assume*

$$\phi(u)_t,\ \phi(v)_t \in L^1(0, T; L^1(\Omega)) \quad \text{and} \quad f - g \in L^1(0, T; L^1(\Omega)), \qquad (4.6)$$

Additionally assume that

$$\begin{aligned} v_0 \leqslant u_0 \quad &\text{a.e. in } \Omega,\\ g \leqslant f \quad &\text{a.e. in } (0, T) \times \Omega, \end{aligned} \qquad (4.7)$$

then $v \leqslant u$ *a.e. in* $(0, T) \times \Omega$.

Proof Use Theorem 4.1 with $\lambda(s) = s^-$ to deduce that

$$\int_\Omega (\phi(u) - \phi(v))^- \leqslant 0,$$

thus, $\phi(u) - \phi(v) \geqslant 0$ *a.e.* in $(0, T) \times \Omega$. Since $\phi(s)$ is strictly increasing the result of the corollary follows. □

Remark 4.3 [Non-negativity for ϕ_{reg} and ϕ] Note that any solution u of problem (1.8) associated to a ϕ_{reg}, with $u_0 \geqslant 0$ and $f \geqslant 0$, is unique and non-negative. The previous observations are consequences of Corollaries 4.1 and 4.2, and the fact that $\phi_{\text{reg}}(u)_t \in L^1(0, T; L^1(\Omega))$. Furthermore, the solution constructed in Step 5 of the proof of existence will be non-negative as well since it is a pointwise limit of solutions associated to regularized problems.

Corollary 4.3 *Assume u and v are weak solutions of problem (1.9) found as in Corollary 2.1, associated to the initial data u_0 and v_0, and the forcing terms f and g, respectively. Assume the additional property that*

$$u_t,\ v_t \in L^1(0, T; L^1(\Omega)) \quad \textit{and} \quad f - g \in L^1(0, T; L^1(\Omega)), \tag{4.8}$$

then

$$\int_\Omega \lambda(u - v) \leqslant \int_\Omega \lambda(u_0 - v_0) + \int_0^t \int_\Omega \lambda(f - g), \tag{4.9}$$

where $\lambda(s)$ is any of the following three functions, $|s|$, s^+ or s^-.

The proof of Corollary 4.3 is an immediate consequence of Theorem 4.1 and it is an equivalent comparison result for problem (1.9). From this corollary, we obtain equivalent uniqueness and non-negativity results for problem (1.9).

5 Open problem: Topographic effects

To the best of our knowledge, existence, uniqueness and regularity of solutions of the DSW equation in its general form (1.1), i.e. when *topographic* effects are considered, have not been studied. Observe that when we formally carry out the spatial differentiation inside the divergence term in the first equation,

$$\frac{\partial u}{\partial t} - h_1(u, z)\, \nabla(u - z) \cdot \nabla u \ - \ h_2(u, z)\, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|^{1-\gamma}} \right) = f,$$

where $h_1(u, z) = \alpha(u - z)^{\alpha-1} / \, |\nabla u|^{1-\gamma}$ and $h_2(u, z) = (u - z)^\alpha$, we can see the appearance of a non-linear advection term, and a non-linear diffusive term involving the bathymetry. The topographic effects change qualitatively the direction of the advection $\nabla(u - z)$, and scale both the advection and the diffusion terms. Some of the difficulties that arise when one introduces a non-flat bathymetry z are

- The aforementioned techniques to prove existence of solutions (used when $z = 0$) fail, since one cannot send the non-linearity $(u - z)^{\alpha}$ to the time-derivative term. In other words the change of variables described at the beginning of section 1.2 does not work correctly. This situation introduces further difficulties when trying to prove the validity of the Galerkin method as a suitable way to obtain approximate solutions.
- In general we expect the regularity of solutions of problem (1.1) to depend on the properties of z. Technically speaking, it is not clear how to proceed in order to incorporate such properties in the analysis and relate them directly with the properties of u.
- Presumably, in order to prove uniqueness of solutions for problem (1.1) we may need to impose an entropy condition as described in [9]. This condition may provide means to identify unique physically consistent solutions.

6 Conclusions

In this paper, we presented a study of basic properties of non-negative solutions for the diffusive wave approximation of the shallow water equations (DSW) in a hydrological context. In our study we presented proofs of the most relevant results existing in the literature using constructive techniques that directly lead to the implementation of numerical algorithms to obtain approximate solutions. We also introduced to both, the engineering and mathematical communities, the problem that arises when topographic effects are considered (obstacle problem) in the DSW, which is to the best of our knowledge, a new avenue of research in the area of theoretical PDEs.

Important issues to be addressed in future works should include:

- An appropriate study of existence and uniqueness of weak solutions of problem (1.1) when topographic effects are considered ($z \neq 0$).
- Regularity of the free boundary for the two-dimensional case both when $z = 0$ (this would be an extension of the work of Esteban and Vázquez [13]), and $z \neq 0$.
- The connection between the regularity of the bathymetry z and the resulting weak solution of problem (1.1).
- Conditions for which the regularity in the time derivative can be improved as well as conditions for which the pointwise gradient can be bounded (for $z = 0$).

Appendix

Lemma A.1 *The operator $\mathscr{A}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by*

$$\mathscr{A}(x) = \frac{x}{|x|^{1-\gamma}} \tag{A 1}$$

is monotone, i.e., for any $x, y \in \mathbb{R}^n$

$$(\mathscr{A}(x) - \mathscr{A}(y)) \cdot (x - y) \geqslant 0.$$

Proof Define the function $\mathscr{B}(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$\mathscr{B}(x) = |x|^{\gamma+1} \qquad \text{where} \qquad |x| = \left(\sum_{j=1}^{n} x_j^2\right)^{\frac{1}{2}}$$

and note that

$$\frac{\partial}{\partial x_i}|x|^{\gamma+1} = (\gamma+1)|x|^{\gamma-1}x_i \qquad \Longrightarrow \qquad \frac{1}{\gamma+1}\nabla\mathscr{B}(x) = \mathscr{A}(x).$$

Since $\gamma + 1 > 1$, the function $\mathscr{B}(x)$ is strictly convex. The gradient of a convex function is strictly increasing in each and all of its components, thus the result of the lemma holds true. □

Theorem A.1 *(Calculus in abstract space) Let X a Banach space and let $u \in W^{1,p}(0,T;X)$ for some $1 \leqslant p \leqslant \infty$. Then*

(i) $u \in C([0,T];X)$ (after possibly being redefined on a set of measure zero), and

(ii) $u(t_1) = u(t_0) + \int_{t_0}^{t_1} u_t(\tau)d\tau$ for all $0 \leqslant t_0 \leqslant t_1 \leqslant T$.

Proof See [14]. □

Assume that Ω is an open, bounded set, with smooth boundary, and $T > 0$. We have

Theorem A.2 *Let ψ be a real valued, absolutely continuous and monotone function, and let $0 < \eta \leqslant \gamma \leqslant 1$. Assume that*

(i) ψ is an η-Hölder continuous function with $\psi(0) = 0$.

(ii) $u_\mu \rightharpoonup u$ in $L^{1+\gamma}(0,T;W^{1,1+\gamma}(\Omega))$.

(iii) $\psi(u_\mu)_t \rightharpoonup v$ in $L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega))$.

Then, $v = \psi(u)_t$.

Proof During the proof every subsequence obtained by a compact argument will be relabelled with the index μ for clarity. Set $p = \frac{1+\gamma}{\eta}$ and note that by (i) and (ii) we have

$$\|\psi(u_\mu)\|^p_{L^p(0,T;L^p(\Omega))} \leqslant \|u_\mu\|^{1+\gamma}_{L^{(1+\gamma)}(0,T;L^{(1+\gamma)}(\Omega))} \leqslant C. \tag{A 2}$$

Since $L^p(0,T;L^p(\Omega))$ is a separable and reflexive Banach space, inequality (A 2) implies that

$$\psi(u_\mu) \rightharpoonup \xi \quad \text{weakly in} \quad L^p(0,T;L^p(\Omega)). \tag{A 3}$$

Since

$$\frac{1+\gamma}{\eta} \geqslant \frac{1+\gamma}{\gamma} = (1+\gamma)^*,$$

it follows that

$$L^p(0,T;L^p(\Omega)) \subset L^{(1+\gamma)^*}(0,T;L^{(1+\gamma)^*}(\Omega)) \subset L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)).$$

Then, for any $\varphi \in C_c^1(0,T)$ and $\omega \in W^{1,1+\gamma}(\Omega)$ we obtain

$$\int_0^T \langle \psi(u_\mu)_t, \varphi\,\omega\rangle = -\int_0^T (\psi(u_\mu), \varphi_t\,\omega).$$

Take $\mu \to \infty$ in this equality to obtain

$$\int_0^T \langle v, \varphi\,\omega\rangle = -\int_0^T (\xi, \varphi_t\,\omega). \tag{A 4}$$

Thus, it remains to prove that $\xi = \psi(u)$.

To this end, we will first prove this for a $\psi' \in L^\infty(\mathbb{R})$. Observe that as a consequence of the chain rule, the sequence

$$\{\psi(u_\mu)\} \subset L^{1+\gamma}(0,T;W^{1,1+\gamma}(\Omega))$$

is uniformly bounded due to (A 2) and (ii). Since the sequence

$$\{\psi(u_\mu)_t\} \subset L^{(1+\gamma)^*}(0,T;W^{-1,(1+\gamma)^*}(\Omega)) \subset L^{1+\gamma}(0,T;W^{-1,1+\gamma}(\Omega))$$

is uniformly bounded by assumption (iii), and

$$L^{1+\gamma}(0,T;W^{1,1+\gamma}(\Omega)) \subset L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega)) \subset L^{1+\gamma}(0,T;W^{-1,1+\gamma}(\Omega))$$

with the compact embedding

$$W^{1,1+\gamma}(\Omega) \hookrightarrow L^{1+\gamma}(\Omega).$$

Thus, we conclude by a compactness criterion in the spaces $L^p(0,T;X)$ that

$$\psi(u_\mu) \to \xi \quad \text{strongly in} \quad L^{1+\gamma}(0,T;L^{1+\gamma}(\Omega)), \tag{A 5}$$

see [21]. This convergence is a.e. in $(0,T)\times\Omega$ as well. Since ψ is invertible, u_μ converges a.e. in $(0,T)\times\Omega$ to $\psi^{-1}(\xi)$. Using (ii) we conclude that

$$u = \psi^{-1}(\xi). \tag{A 6}$$

Now we proceed to extend the previous result for a general ψ having the conditions stated in the hypothesis of the theorem. Let $\{\psi_\epsilon\}$ be a family of absolutely continuous and increasing functions with bounded derivative (for fix $\epsilon > 0$) and fulfilling condition (i), such that for some $k > 0$

$$\sup_s |\psi(s) - \psi_\epsilon(s)| \leqslant k\epsilon^\eta.$$

Hence for any $\varphi \in C_c^1(0,T)$ and $\omega \in W^{1,1+\gamma}(\Omega)$,

$$\begin{aligned}\left|\int_0^T \langle \psi(u_\mu)_t - \psi(u)_t, \varphi\, \omega\rangle\right| &= \left|\int_0^T (\psi(u_\mu) - \psi(u), \varphi_t\, \omega)\right| \\ &= \left|\int_0^T (\psi(u_\mu) - \psi_\epsilon(u_\mu) + \psi_\epsilon(u_\mu) - \psi(u), \varphi_t\, \omega)\right| \\ &\leqslant k\epsilon^\eta \int_0^T (1, |\varphi_t\, \omega|) + \left|\int_0^T (\psi_\epsilon(u_\mu) - \psi(u), \varphi_t\, \omega)\right|. \qquad \text{(A 7)}\end{aligned}$$

Similarly,

$$\left|\int_0^T (\psi_\epsilon(u_\mu) - \psi(u), \varphi_t\, \omega)\right| \leqslant k\epsilon^\eta \int_0^T (1, |\varphi_t\, \omega|) + \left|\int_0^T (\psi_\epsilon(u_\mu) - \psi_\epsilon(u), \varphi_t\, \omega)\right|. \qquad \text{(A 8)}$$

Using (A 7) and (A 8) we take $\mu \to \infty$ to obtain

$$\limsup_\mu \left|\int_0^T \langle \psi(u_\mu)_t - \psi(u)_t, \varphi\, \omega\rangle\right| \leqslant 2k\epsilon^\eta \int_0^T (1, |\varphi_t\, \omega|).$$

Finally, let $\epsilon \to 0$ to conclude. □

Theorem A.3 *Assume that Ω is measurable and $|\Omega| < \infty$. Assume also that $f \in L^p(\Omega)$ for any $1 \leqslant p < \infty$ and $\|f\|_{L^p(\Omega)} \leqslant M$ for some $M > 0$. Then*

$$f \in L^\infty(\Omega) \qquad \text{and} \qquad \|f\|_{L^\infty(\Omega)} \leqslant M. \qquad (6.9)$$

Proof See [32, p. 126] for a version of this result. A slight modification of this proof will work for this version. □

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