Heavy Quark Effective Field Theory*†

Howard Georgi
Lyman Laboratory of Physics, Harvard University
Cambridge, MA 02138

Abstract

In these three lectures, I review the ideas behind heavy quark effective field theory.

Introduction and History

. . . And what there is to conquer
By strength and submission, has already been discovered
Once or twice, or several times, by men whom one cannot hope
To emulate — . . .

East Coker
T.S. Eliot

There is a long list of people who are better qualified than I am to talk to you about this subject. However, when this school was being organized, all of them were too quick for Keith Ellis, so you are stuck with me. I am afraid that I am not very good at giving review talks. For one thing, I lack the patience to do a thorough scouring of the literature. But even if I could, I would probably end up with a very idiosyncratic view of the field. I apologize in advance to all of my colleagues whose contributions I will overlook or misrepresent in the following lectures. Please remember that it is Keith's fault for inviting me. To you students, I will present my personal view of what is going on in this subject without apology or remorse. What's the sense in giving lectures at all if you don't have something unique and personal to say?

My plan in these lectures is as follows. I will concentrate on one rather small area, the matrix elements of weak currents between heavy meson states, because this is the area that I understand best, and that I think is best understood in general. This is important not only for QCD, but also for flavor physics. Ultimately, the applications of the ideas I discuss here will help us to interpret experimental results and pin down the KM matrix. But I will concentrate just on the QCD.

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- I will begin with a brief review of the early history and literature of the subject.
- I will spend most of the rest of the first lecture waving my hands, building up the physical picture of a heavy quark bound state.
- At the end of the first lecture I will briefly discuss the notion of effective field theories in general,and at the beginning of the second, I will discuss at length the construction of the heavy quark effective field theory, and identify the peculiar $SU(2f)^{\infty} \otimes$ Lorentz symmetry of this effective theory.
- In the second lecture, I will show how to use this symmetry structure in a tensor analysis to extract information from the effective field theory efficiently.
- In the last lecture, I will use the effective field theory formalism to relate currents in the full high energy theory to operators in the heavy quark effective field, by "matching and running".

I got interested in the subject of heavy quarks a couple of years ago in my capacity as an Editor of Physics Letters **B**. I received, in quick succession, two interesting papers by Nathan Isgur and Mark Wise [1] about the matrix elements of currents between heavy meson states, in which they discussed a new symmetry of the QCD interactions. Now symmetries of quantum field theory are one of my life-long interests, so I set about trying to understand what Mark and Nathan were talking about.

As it happened ("as it was supposed to happen" as Bokonen would say in Kurt Vonnegut's **Cat's Cradle**), at about the same time I received a paper on a related subject by Esty Eichten and Brian Hill. [2] My own contribution to the subject was to put these two sets of papers together into something that I could understand and use.

But let's start at the beginning. There are two reasons why a heavy quark might be a much simpler thing to think about than a light quark. One is the asymptotic freedom of the QCD coupling. If the quark is sufficiently heavy, then the QCD coupling relevant at a distance scale of the order of the quark's Compton wavelength is small. That should make it's interactions easier to understand. This is an old idea, going back at least as far as the prediction of Charmonium states by Appelquist and Politzer [3] before the discovery of the J/ψ. If all the quarks in the world were sufficiently heavy (very heavy indeed, see section 3 of [4]), this would be the whole story. We would just calculate heavy meson properties from first principles. QCD, at least for the quark states,[1] would be like atomic physics. But the light quarks make the world much more complicated. Asymptotic freedom is not enough to help with our understanding of a bound state of a heavy quark and a light antiquark in a heavy meson or of a heavy quark with two light quarks in a heavy baryon. Here the size of the state is determined by the QCD confinement scale, so that the QCD interactions

[1] The glueballs would still be complicated, but no one would care very much.

of the light constituents are very complicated — at least sufficiently complicated to be confining.

In this first lecture, I am going to use a technique that I learned from David Politzer, that he called “the method of the virtual guru”. The idea is that after you have been the physics business for awhile, you learn that different good physicists have very different skills, and that it sometimes helps to try to adopt the mind set of a virtual guru, and approach the problem the way the guru would. For a little while, in this lecture, my virtual guru is going to be Nathan Isgur, because he explains what is really going in the heavy quark business better than anyone else.

Nathan likes to call the complicated structure of confining QCD associated with the light antiquark in a heavy meson (or light quarks in a heavy baryon) the “brown muck” of hadron physics. I'll adopt this phrase, because it is a nice reminder of the difficulties associated with the strong QCD interactions. When you have to descend into the brown muck, you abandon all pretense of doing elegant, pristine, first-principles calculations. You have to get your hands dirty with uncontrolled approximations and models. When you are finished with the brown muck, you should wash your hands.

In this context, the heaviness of the heavy quark is important for a different reason. As the heavy quark becomes heavier and heavier, you must go in to smaller and smaller distances to see the details of its structure. The color charge of the quark remains as obvious as ever because the color flux extends out to long distances independent of the mass. But relativistic effects such as color magnetism go to zero as the quark mass goes to infinity. But it is only through these relativistic effects that the quark spin couples to the brown muck of the rest of the strongly interacting system. Thus as the quark mass goes to infinity, its spin decouples.

Perhaps the first application of these ideas to brown mucky systems occurred in the early days of the QCD quark model of the strong interactions, after the discovery of the J/ψ, but before the discovery of charmed particles. On the basis of a model incorporating the decoupling of the heavy quark spin, the mass splitting between the D^* and D mesons was predicted to be much less than the $\rho - \pi$ splitting, and estimated to be of the order of m_π, [5] which of course turned out to be about right. Since then, such models of heavy quark systems and their brown muck have been refined by many brave souls. [6, 7][2]

It was also realized that this decoupling of the heavy quark spin could be justified rigorously in QCD by going to the nonrelativistic limit of the heavy quark system. [8, 9] Many of the works on this subject were closely tied to thoughts about lattice QCD. You will hear more about this connection in the following set of lectures by Esty Eichten.

Finally, Voloshin and Shifman [10] and Politzer and Wise [11] understood the effect of QCD renormalization on operators involving a heavy quark, taking into account

[2]We will have more to say about reference [6] below, because while it is largely devoted to discussions of the brown muck, it is one of the first papers to describe one of the essential physical ideas in the subject of heavy quark physics.

the decoupling of the spin and other relativistic effects. Also, in reference [6] and [10], the crucial physical argument (that I will discuss in a moment) is enunciated that allows direct comparisons of bounds states of heavy quarks with different masses.

In principle, one can extract all of heavy quark physics using the techniques of references [10] and [11]. However, in practice, the language used in these papers was rather cumbersome, and no one quite realized how far it could be pushed. The great contribution of Isgur and Wise [1] was to elevate the physical argument of [6] and [10] to the status of a symmetry argument. That, finally, made it possible for people like me to understand what was going on. A couple of years after the original Isgur and Wise papers, their new approach to heavy quark physics has become part of the Canon of perturbative QCD.

Now having breezed through the history, we will forget it. I will certainly not try to reproduce the tortuous logic of history. I will not even always refer back to the original papers. I will simply start with the modern language, and only occasionally refer to older ideas.

The Physical Picture

Pay no attention to Caesar. Caesar has no idea what is really going on.

Cat's Cradle
Kurt Vonnegut

I am interested in the matrix elements of heavy quark currents between heavy meson states. In particular, I will assume that the b and c quarks are heavy enough to make the techniques I discuss useful. This is a questionable assumption, at best, for the c quark, and by no means completely obvious for the b. However, it is what I am going to assume.

Let's begin by considering the bound states of a c quark with a light antiquark, the $J^P = 0^-$ D mesons and $J^P = 1^-$ D^* mesons:

$$\begin{gathered} D^0 \text{ and } D^{*0} = c\overline{u}\,, \quad D^+ \text{ and } D^{*+} = c\overline{d}\,, \\ D_s \text{ and } D_s^* = c\overline{s}\,. \end{gathered} \tag{1.1}$$

The splitting between the spin 0 and spin 1 meson states is small because it is a relativistic effect of the color magnetic interaction, suppressed by $1/m_c$. Furthermore, the decoupling of the heavy quark spin means that in the limit $m_c \to \infty$, the structure of the brown muck in the 0^- and 1^- states is identical. There are 4 states for each flavor of light quark with exactly the same brown muck, the 0^- state and the 3 spin states of the 1^-

The Hilbert space of these 4 states can be conveniently represented in a tensor product notation:

$$|D, \pm 1/2, \pm 1/2\rangle \tag{1.2}$$

where the first $\pm 1/2$ is the z component of the spin of the heavy quark and the second $\pm 1/2$ is the z component of the angular momentum of the brown muck (including the light antiquark). This notation should be very familiar from the example of addition of spin and orbital angular momentum in nonrelativistic quantum mechanics.

We can find the eigenstates of the total angular momentum by the usual Clebsch-Gordan decomposition procedure. In this notation, the meson states are

$$\begin{aligned} |D^*, +1\rangle &= |D, +1/2, +1/2\rangle \\ |D^*, 0\rangle &= \frac{1}{\sqrt{2}} \left(|D, +1/2, -1/2\rangle + |D, -1/2, +1/2\rangle \right) \\ |D^*, -1\rangle &= |D, -1/2, -1/2\rangle \\ |D\rangle &= \frac{1}{\sqrt{2}} \left(|D, +1/2, -1/2\rangle - |D, -1/2, +1/2\rangle \right) \end{aligned} \tag{1.3}$$

Note that the phase relation between the D^* states and the D state is arbitrary, but everything else is completely fixed just by the angular momentum structure.

Now let us consider the matrix elements of currents between these states. The simplest thing to consider is the forward matrix element of the vector current,

$$j^\mu = \bar{c} \gamma^\mu c \tag{1.4}$$

We know that the space integral of this conserved current is the charge that counts the number of c quarks (minus the number of $\bar{c}$ antiquarks, but there aren't any in this problem). With both states at rest, the momentum transfer vanishes and the current's matrix element is determined — the result is

$$\langle D, s_h', s_m' | \, j^\mu \, | D, s_h, s_m \rangle = 2 m_D \, \delta_{s_h' s_h} \delta_{s_m' s_m} \tag{1.5}$$

Although (1.5) is an exact result of a symmetry argument, it will help us to have a physical picture of the result that we can generalize to less trivial situations. Physically, what is happening here is that for very large quark mass, the heavy quark is carrying almost all of the momentum of the meson state. The heavy quark is just barreling along its world line, in this case sitting still and just evolving in time, almost unaffected by the cloud of muck that it carries with it. Its wave function is essentially that of a free heavy quark at rest just because of the kinematics. The "wave function" of a D state can be approximated as a product of the free heavy quark wave function and the complicated wave function that describes the brown muck:

$$|D, s_h, s_m\rangle \approx |c, s_h\rangle \, |\mathrm{muck}, s_m\rangle \tag{1.6}$$

This factorization becomes exact in the limit that the heavy quark mass goes to infinity.

The current acts on the free quark wave function. Again, this is only an approximate result, as we will discuss below. The only nontrivial part of the matrix element of the current between D meson states is an overlap between mucks:

$$\begin{aligned}
&\langle D, s_h', s_m' |\, j^\mu \,| D, s_h, s_m \rangle \\
&\approx \langle c, s_h' |\, j^\mu \,| c, s_h \rangle \, \langle \text{muck}, s_m' \,| \text{muck}, s_m \rangle \\
&= 2m_D \, \delta_{s_h' s_h} \, \langle \text{muck}, s_m' \,| \text{muck}, s_m \rangle
\end{aligned} \tag{1.7}$$

In this case, the muck states are exactly the same, so the overlap integral is trivial,

$$\langle \text{muck}, s_m' \,| \text{muck}, s_m \rangle = \delta_{s_m' s_m} \tag{1.8}$$

In this picture, it is easy to see why we can compute, at least approximately, the matrix elements of not just j^μ, but also of the axial vector current, $j_5^\mu = \overline{c}\gamma^\mu\gamma_5 c$, or in general, any operator of the form $\overline{c}\,\Gamma\, c$, where Γ is a Dirac matrix. The overlap of the muck is still trivial. The matrix element of the current is given approximately by the matrix element between free c quark states:

$$\begin{aligned}
&\langle D, s_h', s_m' |\, \overline{c}\,\Gamma\, c \,| D, s_h, s_m \rangle \\
&\approx \langle c, s_h' |\, \overline{c}\,\Gamma\, c \,| c, s_h \rangle \, \langle \text{muck}, s_m' \,| \text{muck}, s_m \rangle \\
&= \langle c, s_h' |\, \overline{c}\,\Gamma\, c \,| c, s_h \rangle \, \delta_{s_m' s_m}
\end{aligned} \tag{1.9}$$

This describes all the matrix elements of the arbitrary current between all combinations of D and D^* states, so there is a great deal of information here. We will work out some explicit examples of (1.9) later, when have developed the theoretical machinery required to do it most efficiently. What I am interested in getting across now is the physical idea. The important thing is that (1.9) does not depend at all on the details of the brown muck, only that it is the same muck on both sides.

Usually, when we can compute anything reliably in the nonperturbative regime of a strongly interacting theory, there is some symmetry at work, and (1.9) is no exception. When the quark mass goes to infinity, the theory describing the heavy quark states at rest has an extra symmetry because of the decoupling of the heavy quark spin. It is this symmetry that assures us that the brown muck is the same on both sides of (1.9). As we will see later, we will be able to interpret (1.9) as a direct consequence of this spin symmetry, just as (1.5) follows from c-number conservation.

All this is not very surprising, perhaps. What seems much more bizarre at first is that these same considerations can be extended to matrix elements between states with different heavy quarks.

Consider now the bound states of the b quark with a light antiquark, the $J^P = 0^-$ $\overline{B}$ mesons and the $J^P = 1^-$ $\overline{B}*$ mesons:

$$\overline{B}^- \text{ and } \overline{B}^{*-} = b\overline{u}\,, \quad \overline{B}^0 \text{ and } \overline{B}^{*0} = b\overline{d}\,, \quad \overline{B}_s \text{ and } \overline{B}_s^* = b\overline{s}\,, \tag{1.10}$$

and the matrix element of a current of the form $\bar{c}\,\Gamma\,b$ between a $\overline{B}$ and a D or $D*$ state:

$$\langle D|\,\bar{c}\,\Gamma\,b\left|\overline{B}\right\rangle\ ,\quad \langle D^*|\,\bar{c}\,\Gamma\,b\left|\overline{B}\right\rangle\ . \tag{1.11}$$

These matrix elements are particularly interesting because for $\Gamma=\gamma^\mu$ and $\Gamma=\gamma^\mu\gamma_5$, they are matrix elements relevant to the semileptonic weak decay of the $\overline{B}$ meson into D or D^* plus leptons.

As for the D states, we can label the $\overline{B}$ states by the heavy quark spin and the angular momentum of the brown muck,

$$\begin{gathered}
\left|\overline{B}^*,+1\right\rangle=\sqrt{2m_B}\left|\overline{B},+1/2,+1/2\right\rangle \\
\left|\overline{B}^*,0\right\rangle=\sqrt{2m_B}\frac{1}{\sqrt{2}}\left(\left|\overline{B},+1/2,-1/2\right\rangle+\left|\overline{B},-1/2,+1/2\right\rangle\right) \\
\left|\overline{B}^*,-1\right\rangle=\sqrt{2m_B}\left|\overline{B},-1/2,-1/2\right\rangle \\
\left|\overline{B}\right\rangle=\sqrt{2m_B}\frac{1}{\sqrt{2}}\left(\left|\overline{B},+1/2,-1/2\right\rangle-\left|\overline{B},-1/2,+1/2\right\rangle\right)
\end{gathered} \tag{1.12}$$

and then approximately factor them into free heavy quark wave functions and brown muck wave functions:

$$\left|\overline{B},s_h,s_m\right\rangle\approx|b,s_h\rangle\,|\text{muck},s_m\rangle \tag{1.13}$$

Again the factorization becomes exact in the limit that the heavy quark mass goes to infinity.

Again, as the quark mass goes to infinity, the current acts on the free quark wave function, so the matrix element looks analogous to (1.9),

$$\begin{aligned}
&\langle D,s_h',s_m'|\,\bar{c}\,\Gamma\,b\left|\overline{B},s_h,s_m\right\rangle \\
&\approx\langle c,s_h'|\,\bar{c}\,\Gamma\,b\,|b,s_h\rangle\,\langle\text{muck},s_m'\,|\text{muck},s_m\rangle \\
&\approx\langle c,s_h'|\,\bar{c}\,\Gamma\,b\,|b,s_h\rangle\,\delta_{s_m's_m}
\end{aligned} \tag{1.14}$$

There are several things to note about (1.14).

1. Again the point is not that we know anything about the muck wave functions, but only that they are more or less the same on both sides, in the D state and in the $\overline{B}$ state. Once the quark is sufficiently heavy, it just sits in the middle of the bound states and produces a static color Coulomb field.

2. The relation, (1.14), like the relation, (1.9), can be interpreted in terms of a symmetry of the theory describing the heavy quark states as the quark masses go to infinity. But it is a rather odd symmetry in that it relates the D state and the $\overline{B}$ state with different 4-momenta. What matters, in this symmetry, is not the 4-momentum, but the 4-velocity. The brown muck looks the same in

the D and $\overline{B}$ states when both heavy quarks are at rest. Likewise the brown muck in a D state moving with 4-velocity v looks the same as the brown muck in a $\overline{B}$ state moving with 4-velocity v.

3. The matrix element, (1.14), is not a forward matrix element with zero momentum transfer to the current like that in (1.9). Instead, because the symmetry relates states with the same velocity, the calculable matrix element involves the maximum possible momentum transfer. [6, 10] The kinematic point where the velocities are equal is sometimes referred to as the Shifman-Voloshin point.

This discussions suggests that we should be able to say something about matrix elements between states of different momenta. Of course, we should label the states not by their momenta, but by their velocities. But the physical picture is much the same. For large quark masses, the matrix elements should factor into a piece that involves the heavy quarks and is more or less determined kinematically, and a piece that is an overlap integral between the brown mucks in the initial and final states.

$$
\begin{aligned}
&\langle D, s_h', s_m', v' |\, \bar{c}\,\Gamma\, c\,| D, s_h, s_m, v\rangle \\
&\approx \langle c, s_h', v' |\, \bar{c}\,\Gamma\, c\,| c, s_h, v\rangle\, \langle \text{muck}, s_m', v'\, | \text{muck}, s_m, v\rangle \\
&= \langle c, s_h', v' |\, \bar{c}\,\Gamma\, c\,| c, s_h, v\rangle\, \xi_{s_m' s_m}(v', v)
\end{aligned}
\tag{1.15}
$$

One might think that this overlap integral, $\xi_{s_m' s_m}(v', v)$, could be a complicated matrix function in the spin space of the brown muck, involving many unknown functions. But in fact, it follows from Lorentz invariance and parity invariance of the QCD interactions that it depends on only a single function, $\xi(v'v)$.

It is easiest to see this in the brick will frame.

$$\vec{v'} = -\vec{v}\,, \quad v'^0 = v^0 = \sqrt{1 + |\vec{v}|^2} \tag{1.16}$$

$$v'v = v'^0 v^0 + |\vec{v}|^2 = 1 + 2\,|\vec{v}|^2 \tag{1.17}$$

$$|\vec{v}|^2 = \frac{v'v - 1}{2} \tag{1.18}$$

In this frame, angular momentum around the $\vec{v}$ direction is conserved. But because of the decoupling of the heavy quarks spins, the angular momentum of the brown muck is separately conserved. When the external current turns the heavy quark around, it exerts no torque on the brown muck because of decoupling. Therefore the helicity of the incoming brown muck is opposite to that of the outgoing brown muck.

$$\xi_{s_m' s_m}(v', v) = \delta_{s_m', -s_m}\, \xi_{s_m}(v'v) \tag{1.19}$$

Then parity invariance implies that the overlap of the brown muck is the same for incoming left handed muck, and incoming right handed muck.

$$\xi_{1/2}(v'v) = \xi_{-1/2}(v'v) \equiv \xi(v'v) \tag{1.20}$$

so

$$\xi_{s'_m s_m}(v', v) = \delta_{s'_m, -s_m}\, \xi(v'v) \tag{1.21}$$

I call this $\xi(v'v)$ the "Isgur-Wise" function. The Isgur-Wise function summarizes the real nonperturbative dynamics. We can't calculate it. But even so, this picture is an enormous help in reducing the large number of independent functions required to describe the process in general, to just one in the heavy quark limit. We will see later how to compute the matrix structure very simply.

The same argument applies equally well when the current changes the heavy quark flavor. The matrix elements

$$\begin{aligned}
&\langle D, s'_h, s'_m, v' |\, \bar{c}\, \Gamma\, b \left| \overline{B}, s_h, s_m, v \right\rangle \\
&\approx \langle c, s'_h, v' |\, \bar{c}\, \Gamma\, b \,| b, s_h, v\rangle \, \langle \text{muck}, s'_m, v' \,| \text{muck}, s_m, v\rangle \\
&\approx \langle c, s'_h, v' |\, \bar{c}\, \Gamma\, b \,| b, s_h, v\rangle \, \xi_{s'_m s_m}(v', v)
\end{aligned} \tag{1.22}$$

should factor into a piece that involves the heavy quarks and is determined more or less determined kinematically, and a piece that is an overlap integral between the brown mucks in the initial and final states. The brown muck doesn't know that the heavy quark flavor has changed! There is still a tiny source of color charge in the middle, and that is all the brown muck knows about.

In fact, we will discover that there are important corrections to these relations from the QCD interactions. This is because the approximate factorization of the matrix element depends on the renormalization scale in QCD.

Effective Field Theories

Sufficient unto the day is the evil thereof.

Mt. 6:34
King James Version

The idea of effective field theories comes straight from the Sermon on the Mount (which is well know to historians of science as an early warning against the excesses of string theory). The point is a simple one, but very important. If you want to understand what is happening in some physical process, it is counterproductive to have to determine how the process fits into a theory of everything (even assuming that such a concept makes sense). Instead, one should use a level of description that is well matched to the problem at hand.

In QCD, the relevant variable is the "momentum scale". The generic effective field theory analysis looks like this:

Large Scale
ϕ_j, χ

renormalization group

$\mu = \Lambda$
particle mass

MATCHING

renormalization group

Low Energy
ϕ_j

The process of matching the two theories at a heavy particle threshold is sometimes described as "integrating out" the heavy particle. But in fact, the process is somewhat more involved. When the heavy particle is integrated out of the theory, what results is a non-local action. An extra step is required to get to the effective Lagrangian in the low energy theory. One must disentangle the short distance physics that is incorporated into the coefficients of the effective Lagrangian from the long distance physics that remains explicit in the low energy theory. It is here that "matching" really comes in.

The way this disentangling works is instructive. Matching corrections are computed by comparing calculations of physical quantities in the high energy theory ($\mu \geq \Lambda$) with calculations of the same quantities in the low energy theory ($\mu < \Lambda$) and choosing the parameters in the effective theory so that the physics is the same at the boundary ($\mu = \Lambda$). Any interaction that is unchanged in the matching cancels out of the matching because it contributes in the same way in the two theories. The **change** in a parameter in the effective theory due to matching is related, order by order in perturbation theory, to a difference between the high energy and low energy calculations. In this difference, all effects of long distance physics, infrared divergences, physical cuts, *etc.*, disappear, because they are the same, by construction, in the high and low energy theories. Thus only the short distance contributions are incorporated into the coefficients of the effective Lagrangian.

Heavy Quarks and the Velocity Superselection Rule

> ... *strait is the gate, and narrow is the way,* ...
>
> **Mt. 7:14**
> King James Version

The question is, what do you do if you are stuck at low energies (this often happens because of budgetary constraints — the taxpayers sometimes won't build you a big enough accelerator) but your universe contains a stable very heavy particle? This things just lumbers along through your universe carrying a very large momentum. Because it's momentum is huge, its evolution is essentially classical. What does the effective field theory look like?

In the heavy quark example, the low energy scale I have in mind is $\Lambda_{\rm QCD}$, the typical energy scale of the brown muck. A quark is heavy if $m_q \gg \Lambda_{\rm QCD}$.

In these lectures, I will stick to the simple situation in which you have only one heavy quark in your universe at any given time. If you have more than one, life gets much more difficult because the two heavy quarks can exchange gluons that carry large momenta.

However, we can imagine that the heavy quark changes its velocity in response to some force that does not involve the QCD interactions. This does not pump any energy into the light quarks and gluons, so we stay at the energy scale of the brown muck. Similarly, we can consider transitions between one heavy quark and another. No more energy is pumped into the brown muck when a b quark with velocity v is kicked by a weak current into a c quark with velocity v', than when a c quark with velocity v is kicked by an electromagnetic current into a c quark with velocity v'. In fact, as we have seen, the brown muck doesn't know the difference.

The external currents are very useful as a way of keeping track of the trajectory of the heavy quark. The external currents introduce kinks into the trajectories.

The question is, what does this effective theory look like? Evidently, the presence of a heavy quark traveling with a definite velocity breaks Lorentz invariance, but we would expect the invariance to be restored when we consider all possible heavy quark velocities. What is the symmetry structure of this peculiar theory?

Consider a heavy quark bound state with velocity v.

$$P^\mu_{\rm bound\ state} = m_{\rm bound\ state} v^\mu \tag{2.23}$$

for large m,

$$m_{\text{quark}} \approx m_{\text{bound state}}$$

We expect the difference to be independent of m_{quark}. We expect the heavy quark to carry most, but not all of the momentum of the bound state. There will be a small momentum, q^μ, of the brown muck from low energy QCD interactions. Then we can write

$$p^\mu_{\text{quark}} = P^\mu_{\text{bound state}} - q^\mu = m_{\text{quark}} v^\mu + k^\mu \tag{2.24}$$

where this defines the small residual momentum

$$k^\mu = (m_{\text{bound state}} - m_{\text{quark}}) v^\mu - q^\mu$$

Now compare the 4-velocity fo the heavy quark with that of the state:

$$v^\mu_{\text{quark}} = \frac{p^\mu_{\text{quark}}}{m_{\text{quark}}} = v^\mu + k^\mu / m_{\text{quark}} \tag{2.25}$$

They are the same in the heavy quark limit.

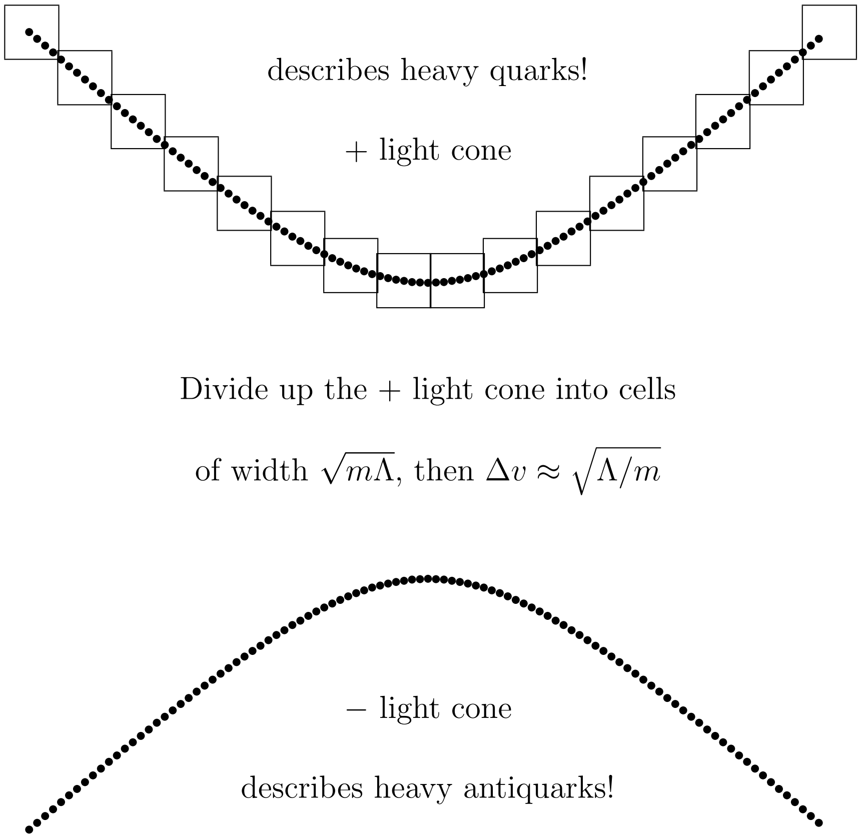
$$v^\mu_{\text{quark}} \to v^\mu \quad \text{as} \quad m_{\text{quark}} \to \infty$$

The QCD interactions do not change heavy quark's velocity, no matter what the brown muck is doing!

This leads to the velocity superselection rule. [12] Under the influence of the QCD interactions in the low energy theory, the heavy quarks move in straight line trajectories. This is just conservation of momentum. If QCD interactions, by assumption, can only change the momentum by a small momentum, the change in the 4-velocity, $v = p/m$, due to the soft QCD interactions is negligible. All the kinks in the trajectories must be caused by some external (**non-QCD**) agency, like a weak or electromagnetic interaction. These are represented by $c^\dagger_{v'} c_v$ type operators that annihilate a heavy quark with velocity v and create a heavy quark with velocity v'.

I will discuss three related (indeed equivalent) ways of looking at the effective theory — first looking at the action in momentum space — next discussing Feynman graphs and the form of the propagator — and finally showing what is happening to the quark fields in position space. The starting point is always the relation $p^\mu = m v^\mu + k^\mu$.

First consider the functional integral in momentum space (the space is shown in the figure below):



The quark kinetic energy term in the action looks like this:

$$\int \frac{d^4p}{(2\pi)^4}\,\overline{c}(-p)\,(\not p - m)\,c(p) \tag{2.26}$$

where $\not a = a_\mu \gamma^\mu$. For a heavy quark of velocity v, the relevant region is $p^\mu = mv^\mu + k^\mu$ for $k^\mu \ll m$ and the only relevant $c(p)$ are those for which p is almost on mass shell, satisfying

$$(\not v - 1)\; c(p) \approx -\frac{\not k}{m}\, c(p) \approx 0 \tag{2.27}$$

Everything else gives a large contribution to the action, proportional to m, and thus it makes a small contribution to the functional integral. In a cell around $p^\mu = mv^\mu$, define the heavy quark field $c_v(k) = c(p) - \mathcal{O}(1/m)$ satisfying exactly

$$\not v\, c_v = c_v \tag{2.28}$$

In other words, I am ignoring the variation of the spinor within the cell, because the effect of this variation will vanish as $m \to \infty$.

Then in terms of the residual momentum, with $k = p - mv$ small compared to m, the Lagrangian in the cell looks like

$$\overline{c_v}\,(\not{p} - m)\,c_v = \overline{c_v}\;\not{k}\,c_v = \overline{c_v}\,v^\mu k_\mu\,c_v \tag{2.29}$$

because

$$\overline{c_v}\,\gamma^\mu\,c_v = \frac{1}{2}\overline{c_v}\,\{\not{v}\,,\gamma^\mu\}\,c_v\overline{c_v}\,v^\mu\,c_v\,. \tag{2.30}$$

Now as $m \to \infty$ and $\Lambda \to \infty$, the cells get closer together in velocity space, but the size of each cell in momentum space also grows. Each cell becomes a mini-Lagrangian relevant only for the heavy quark field with the corresponding velocity.

$$\int_{\text{cell}} \frac{d^4k}{(2\pi)^4}\,\bar{c}_v(-k)\,v^\mu k_\mu\,c_v(k)\,. \tag{2.31}$$

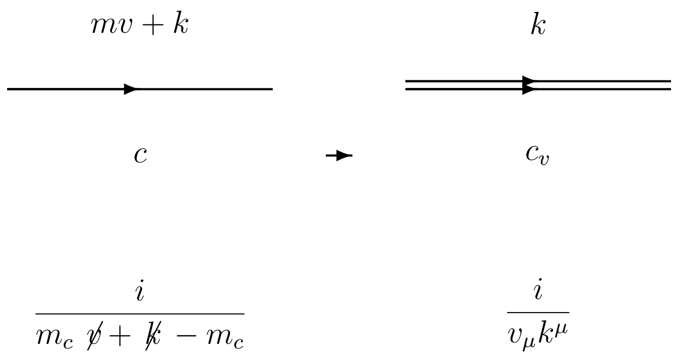
Next look at Feynman propagator in the full theory [13]

$$\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \tag{2.32}$$

If we insert $p^\mu \to mv^\mu + k^\mu$ and take the leading term in m in both the numerator and the denominator

$$\approx \frac{m\;\not{v} + m}{2m(vk) + i\epsilon} = \frac{1+\not{v}}{2}\frac{1}{(vk) + i\epsilon} \tag{2.33}$$

This gives just the form we would expect from the Lagrangian in the cell corresponding to velocity v above. I will denote this heavy quark propagator by a double solid line, to distinguish it from the propagator in the full theory.



Finally consider the Lagrangian in position space. In the full theory, the heavy quark field c has Lagrangian

$$\mathcal{L} = \bar{c}\,(i\,\not{\partial} - m)\,c \tag{2.34}$$

— denote the heavy quark field by c_v. The relation of c_v with the usual heavy quark field c is the following (projecting away the negative light cone which corresponds to the antiquarks):

$$c(x) = \frac{1+\not{v}}{2}\,e^{-im\,v_\mu x^\mu}\,c_v(x) + \mathcal{O}(1/m) \tag{2.35}$$

$$\mathcal{L} \to i\overline{c_v}\,v^\mu \partial_\mu\,c_v \tag{2.36}$$

We have been ignoring the QCD interactions, but it is easy to incorporate them just by imposing the color gauge symmetry.

$$\mathcal{L}_{cv} = i\,\overline{c_v}\,v_\mu D^\mu c_v$$

This gives the same propagator again

$$\frac{1+\not{v}}{2}\frac{1}{(vk)+i\epsilon} \tag{2.37}$$

It is instructive to look at the propagator in position space. In the rest frame the momentum space propagator is

$$\frac{1+\gamma^0}{2}\frac{1}{k^0+i\epsilon} \tag{2.38}$$

The fourier transform is

$$\propto \delta^3\left(\vec{r}-\vec{r'}\right)\,\Theta(t-t') \tag{2.39}$$

This just describes the particle sitting still, propagating in time along its classical trajectory.

The velocity superselection rule is equivalent to the statement that c_v and $c_{v'}$ are **independent fields** for $v^\mu \neq v'^\mu$ — they correspond to different cells on the mass shell hyperboloid.

However, these unrelated independent fields are connected by Lorentz transformations:

$$c_v(x) \to \mathcal{D}(\Lambda)^{-1}\,c_{\Lambda^{-1}v}(\Lambda^{-1}x) \tag{2.40}$$

where $\mathcal{D}(\Lambda) = e^{i\epsilon_{\mu\nu}\sigma^{\mu\nu}}$ is the usual Dirac representation.

The Lagrangian is a sum (not an integral)[3] over all v.

$$\mathcal{L}_c = \sum_{\vec{v}} \mathcal{L}_{cv} \tag{2.41}$$

In this case, instead of integrating out heavy degrees of freedom, we have integrated extra degrees of freedom **IN** to describe the fact that infinitely heavy quarks with

[3] This is written down incorrectly in my original paper on heavy quark effective field theories.

different velocities have infinitely different momenta, and are therefore unrelated to each other so far as the low energy theory is concerned.

$$\mathcal{L}_{cv} = i\,\overline{c_v}\,v_\mu D^\mu c_v$$

Heavy antiquarks, which I will denote by $\underline{c}_v$, can be treated in a similar way. The relation of $\underline{c}_v$ with the usual heavy quark field c is the following (this time projecting away the quarks on the positive light cone):

$$\underline{c}_v(x) = \frac{1-\not{v}}{2}\,e^{-im\,v_\mu x^\mu}\,c(x) \tag{2.42}$$

$$\mathcal{L} \to -i\overline{\underline{c}_v}\,v^\mu\partial_\mu\,\underline{c}_v \tag{2.43}$$

Incorporating the color gauge symmetry gives the Lagrangian

$$\mathcal{L}_{\underline{c}v} = -i\,\overline{\underline{c}_v}\,v_\mu D^\mu \underline{c}_v$$

Despite the $-$, this propagator is really almost the same as the quark propagator if you are describing heavy antiquarks propagating forward in time, because the creation and annihilation operators have been interchanged.[4]

In my original paper on the subject, I put the quark and antiquark Lagrangians together into a single structure using the Dirac algebra. While there is nothing really wrong with this, it is potentially misleading, and I now think that it should be avoided. In the low energy theory, the heavy antiquark has nothing to do with the corresponding heavy quark (at least, in leading order in $1/m$). They live in a completely disconnected region of momentum space, infinitely far away in the heavy quark limit. The form of the Lagrangian should not disguise that fact.

We now will identify the symmetries of this system. It is simplest to see what the symmetry looks like for a heavy quark in its rest system, $v^0 = 1$, $\vec{v} = 0$, for which the Lagrangian is

$$\mathcal{L}_{c0} = i\,\overline{c_0}\,D^0 c_0$$

where

$$\gamma^0 c_0 = c_0 \tag{2.44}$$

This is invariant under rotation in the two dimensional space on which the heavy quark spinor is nonzero. To see what it looks like explicitly, we need a representation for the γ matrices. It is simplest to take γ^0 diagonal, so in a tensor product notation,

$$\vec{\sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \tau_1 \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \tau_2 \equiv \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \tau_3 \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{2.45}$$

we take $\gamma^0 = \tau_3$, $\vec{\gamma} = i\vec{\sigma}\tau_1$. Then the generators of the spin symmetry are

$$\vec{S}_0 = \left(\frac{1+\tau_3}{2}\right)\vec{\sigma} \tag{2.46}$$

[4]As we will discuss below, the difference is only in the structure of the QCD interactions — antiquarks and quarks have different color charges.

Now look at the part of the Lagrangian describing a b quark at rest

$$\mathcal{L}_{b0} = i\,\overline{b_0}\,D^0 b_0$$

$\mathcal{L}_{c0} + \mathcal{L}_{b0}$ has $SU(4)$ symmetry. The reason is that the color interactions of the c_0 and b_0 are just the same. Both just produce a color Coulomb field in the rest frame. To see what the symmetry looks like, put the two field together into an 8-component field:

$$h_0 \equiv \begin{pmatrix} c_0 \\ b_0 \end{pmatrix} \tag{2.47}$$

then

$$\mathcal{L}_{c0} + \mathcal{L}_{b0} = i\,\overline{h_0}\,D^0 h_0 \tag{2.48}$$

In this 8 dimensional space, we need another set of Pauli matrices, $\vec{\eta}$, that implements the $SU(2)$ rotation between c and b subspaces. Then the Isgur-Wise $SU(4)$ spin-flavor symmetry is generated by

$$P_0\sigma_j\,,\ P_0\eta_j,\,,\ P_0\sigma_j\eta_k \ \text{ for } j,k = 1 \text{ to } 3 \tag{2.49}$$

where

$$P_0 = \frac{1+\tau_3}{2} \tag{2.50}$$

Let us now consider the general situation, for v^μ

$$\mathcal{L}_{cv} = i\,\overline{c_v}\,v_\mu D^\mu c_v$$

Remember that because of the superselection rule, c_v is a completely different field from c_0.

We can define “spin” operators for any given v^μ:

$$S_j^v = i\epsilon_{jk\ell}[\not{e}_k, \not{e}_\ell]\,(1+\not{v})/8 \tag{2.51}$$

where e_j^μ for $j = 1$ to 3 is an orthonormal set of space-like vectors orthogonal to v^μ,

$$e_{j\,\mu}e_k^\mu = -\delta_{jk}\,, \qquad v_\mu e_j^\mu = 0 \tag{2.52}$$

The S_j^v have the commutation relations of $SU(2)$. Note that the S_j^0 could be defined in the same way with the $e_j{}^\mu$ being unit vectors along the 1, 2 and 3 axes, $e_j{}^\mu = \delta_j^\mu$.

Now the Lagrangian is invariant under the transformation

$$\delta c_v = i\,\vec{\epsilon}\cdot\vec{S}^v\,c_v \tag{2.53}$$

It follows that

$$\delta\overline{c_v} = -i\,\overline{c_v}\,\vec{\epsilon}\cdot\vec{S}^v \tag{2.54}$$

This looks more complicated than the $v = 0$ that we just discussed, but it really isn't. In fact, we can go to the rest frame by a Lorentz transformation. This is the clue to the symmetry structure of the heavy c quark Lagrangian.

There is one $SU(2)$ for each v. Lorentz transformations change v, and with v, the $SU(2)$'s change as well. I'm not really sure what this symmetry structure is called by mathematicians, but I will denote it by

$$SU(2)^\infty \otimes \text{Lorentz}$$

This is almost like a gauge symmetry, except that the Lorentz transformations also rotate the $SU(2)$'s among themselves, because the e_j^μ rotate under Lorentz transformations.

The b quarks can be added in the same way as in the rest frame. The Lagrangian is

$$\mathcal{L}_{bv} = i\, \overline{b_v}\, v_\mu D^\mu b_v$$

and $\mathcal{L}_{cv} + \mathcal{L}_{bv}$ has an $SU(4)$ symmetry, constructed in the obvious way. If we put the c and b together as before,

$$h_v \equiv \begin{pmatrix} c_v \\ b_v \end{pmatrix} \tag{2.55}$$

then Isgur-Wise $SU(4)$ spin-flavor symmetry is generated by

$$P_v S_j^v\,,\; P_v \eta_j,\,,\; P_v S_j^v \eta_k \;\text{ for } j,k = 1 \text{ to } 3 \tag{2.56}$$

where

$$P_v = \frac{1+ \not{v}}{2} \tag{2.57}$$

So the c-b system together has an

$$SU(4)^\infty \otimes \text{Lorentz}$$

symmetry structure.

Note that the QCD interactions here are important. The heavy quark kinetic energy (the sum over all v) has a much larger symmetry, because we can rotate quarks with different v's into one another. **But this symmetry is broken by the QCD interactions.** For each v, there is a different characteristic color field produced by the heavy quark, just the Coulomb field in the appropriate rest frame.

Likewise, there is no symmetry between heavy quark and heavy antiquark, because the color interactions are different. On the over hand, heavy color triplets with other spins (0, 1, *etc.*) could be included into the heavy quark formalism if Nature is so generous as to give such objects (relatively stable) to play with. The brown muck in a bound state of a heavy triplet scalar or gauge boson looks that same as in a $\overline{B}$ meson! [14, 15, 16]

Tensor Methods

Group theory is a useful technique, but it is no substitute for physics.

Lie Algebras in Particle Physics
Howard Georgi

Sales of my group theory book have been dropping off lately — I thought that I should remind all of you that there is a lot of good pithy stuff there! I next want to develop a tensor analysis to enable us to construct matrix elements consistent with the $SU(2)^\infty \otimes$ Lorentz symmetry in a simple and intuitive way. To do this, we will have to find wave functions of heavy quark states in the effective theory consistent with $SU(2)^\infty \otimes$ Lorentz, and understand how to use them.

Tensor analysis is useful because it makes it easier both to calculate and to understand the results of symmetry arguments.

A very familiar example is Gell-Mann's $SU(3)$. We could get all the information about the consequences of $SU(3)$ symmetry just by clever use of raising and lowering operators. But much easier to manipulate matrices, like

$$\begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & P+ \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & N \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{3}} \end{pmatrix} \tag{2.58}$$

The matrix, (2.58) is a generalized baryon wave function that includes flavor as an additional index. It describes the wave functions of all the octet baryons at once. The symbols are labels for the different baryon wave functions. For example, to describe a proton state, we would set $P \neq 0$ and all the other entries to 0.

Now lets look at the D and D^* states and try to construct something similar. We already know the transformation properties of the fields under the $SU(2)^\infty \otimes$ Lorentz symmetry. Let's use them to construct the states. Define the wave functions as follows:

$$\begin{aligned} D(v) &\propto \langle 0| \, c_v \overline{q} \, |D, v\rangle \\ D(v,\varepsilon) &\propto \langle 0| \, c_v \overline{q} \, |D^*, v, \varepsilon\rangle \end{aligned} \tag{2.59}$$

where ε is the D^* polarization ($\varepsilon^*_\mu \varepsilon^\mu = -1$). $D(v)$ and $D(v,\varepsilon)$ are 4×4 matrix wave functions, analogous to a spinor wave function of a spin-$\frac{1}{2}$ fermion, F,

$$u_F(p) \propto \langle 0| \, \psi \, |F\rangle \ . \tag{2.60}$$

The nice thing about this is that we know its symmetry properties. It has one index that transforms under the $SU(2)^v$ symmetry, the spin of the heavy quark, and it must transform appropriately under Lorentz transformations.

Then the analog of the baryon matrix is the wave function of a general state

$$|\mathbf{D}, v\rangle \equiv d\ |D, v\rangle + \sum_{\varepsilon} d_{\varepsilon}\ |D^*, v, \varepsilon\rangle \tag{2.61}$$

which is

$$\mathbf{D}(v) \propto \langle 0|\, c_v \overline{q}\, |\mathbf{D}, v\rangle = d\, D(v) + \sum_{\varepsilon} d_{\varepsilon}\, D(v, \varepsilon) \tag{2.62}$$

Now d and d_ε are labels, like the baryon labels in (2.58), $d = 1$ in the D state and $d_\varepsilon = 1$ in D^* state with polarization ε.

Because of the properties of c_v, $\mathbf{D}(v)$ satisfies

$$\not{v}\,\mathbf{D}(v) = \mathbf{D}(v) \tag{2.63}$$

and transforms under $SU(2)^v$ as

$$\mathbf{D}(v) \to e^{-i\vec{\epsilon}\cdot\vec{S}^v}\,\mathbf{D}(v) \tag{2.64}$$

The set of all $\mathbf{D}(v)$ must also transform properly under Lorentz transformations:

$$\mathbf{D}(v) \to \mathcal{D}(\Lambda)^{-1}\mathbf{D}(\Lambda^{-1}v)\,\mathcal{D}(\Lambda) \tag{2.65}$$

The transformations (2.64) and (2.65) are the statement, in tensor language, of the $SU(2)^\infty \otimes$ Lorentz symmetry. the $SU(2)^v$ spin symmetry that relates D and D^* is now automatically incorporated by using $\mathbf{D}$ instead of the separate wave functions and constructing invariant matrix elements.

We can now use symmetry arguments to determine $D(v)$ and $D^*(v)$ almost completely.

Lorentz invariance, parity and $\not{v}\,\mathbf{D}(v) = \mathbf{D}(v)$ imply that we can take

$$D(v) = -\sqrt{m_D}\,\frac{1+\not{v}}{2}\,\gamma_5 \tag{2.66}$$

$$D(v, \varepsilon) = \sqrt{m_D}\,\frac{1+\not{v}}{2}\,\not{\varepsilon} \tag{2.67}$$

The overall normalization is completely arbitrary, however, we have the relative normalization correct, up to a conventional phase between the D and D^* states. We have included the factors of $\sqrt{m_D}$ to make the counting of dimensions easier in the applications that follow. Note that $m_D = m_{D^*}$ in the $m \to \infty$ limit, so the dimensional factor is the mass of the state in both (2.66) and (2.67).

To see that the relative normalization is correct, let us look at the norms of the states. These must satisfy (suppressing the momentum δ-function)

$$\begin{aligned} \langle D, v\,|D, v\rangle &= \langle D^*, v, \varepsilon\,|D^*, v, \varepsilon\rangle \\ \langle D^*, v, \varepsilon\,|D, v\rangle &= \langle D, v\,|D^*, v, \varepsilon\rangle = 0 \end{aligned} \tag{2.68}$$

or

$$\langle \mathbf{D}, v \, | \mathbf{D}, v \rangle \propto |d|^2 + \sum_\varepsilon |d_\varepsilon|^2 \tag{2.69}$$

As a first example of the use of tensor methods, let us check that (2.68) and (2.69) actually work. To do this, we have to introduce a little more formalism. The matrix element is proportional to the wave functions put together consistent with the $SU(2)^\infty \otimes$ Lorentz symmetry. We already know that the $|\mathbf{D}, v\rangle$ state has wave function $\mathbf{D}(v)$. The $\langle \mathbf{D}, v|$ state has a wave function proportional to $\mathbf{D}(v)^*$, but this transforms unpleasantly under $SU(2)^\infty \otimes$Lorentz. We are familiar with this difficulty in dealing with fermion wave functions, $u(p)$, where $u(p)^*$ transforms inconveniently, and instead we define a $\overline{u}(p) = u(p)^\dagger \gamma^0$ that is easier to handle. We can do the same thing with the $\mathbf{D}(v)^*$'s. They are, from (2.62),

$$\mathbf{D}(v)^* \propto \langle \mathbf{D}, v| \, (c_v \overline{q})^* \, |0\rangle \tag{2.70}$$

To turn the c_v^* into $\overline{c_v}$ and $(\overline{q})^*$ into q, we must transpose the matrix and multiply on both sides by γ^0. Thus it makes sense to define

$$\overline{\mathbf{D}}(v) \equiv \gamma^0 \mathbf{D}(v)^\dagger \gamma^0 \propto \langle \mathbf{D}, v| \, q \overline{c}_v \, |0\rangle \tag{2.71}$$

This object is acted on on the right by $SU(2)^v$,

$$\overline{\mathbf{D}}(v) \to \overline{\mathbf{D}}(v) \, e^{i\vec{\epsilon}\cdot\vec{S}^v} \tag{2.72}$$

and it transform normally under the Lorentz symmetry,

$$\overline{\mathbf{D}}(v) \to \mathcal{D}(\Lambda)^{-1} \overline{\mathbf{D}}(\Lambda^{-1} v) \, \mathcal{D}(\Lambda) \tag{2.73}$$

We can now compute $\overline{\mathbf{D}}(v)$ from $D(v)$ and $D(v, \varepsilon)$,

$$\overline{\mathbf{D}}(v) \equiv d^* \, \overline{D}(v) + \sum_\varepsilon d_\varepsilon^* \, \overline{D}(v, \varepsilon) \tag{2.74}$$

where because $\gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$ and $\gamma^0 \gamma_5{}^\dagger \gamma^0 = -\gamma_5$,

$$\overline{D}(v) = \sqrt{m_D} \, \gamma_5 \, \frac{1 + \not{v}}{2} \tag{2.75}$$

$$\overline{D}(v, \varepsilon) = \sqrt{m_D} \, \varepsilon_\mu^* \gamma^\mu \, \frac{1 + \not{v}}{2} \tag{2.76}$$

We can now state the rule for computing matrix elements in the low energy theory

$$\langle \mathbf{D}', v' | \, \mathcal{O} \, |\mathbf{D}, v\rangle \tag{2.77}$$

1. Replace the ket $|\mathbf{D}, v\rangle$ by the wave function $\mathbf{D}(v)$;

2. Replace the bra $\langle \mathbf{D}, v|$ by the wave function $\overline{\mathbf{D}}(v)$;

3. Put the wave functions together consistent with the $SU(2)^\infty \otimes$Lorentz symmetry into an object with the same symmetry structure as $\mathcal{O}$. Each independent way of putting things together gets multiplied by an unknown function of all the invariants. Each of these functions must be fixed by the dynamics (or some other argument) in order to completely determine the matrix element.

Example 1 — we can now check (2.69): The heavy quark spin indices must be contracted for $SU(2)^v$ invariance. Then we must form a trace over the remaining Dirac indices with all the possible Lorentz invariant combinations, 1, $\not v$, $\not v\ \not v$, *etc.*. Because $\mathbf{D}(v)\ \not v = -\mathbf{D}(v)$ and $\not v\ \not v = v_\mu v^\mu = 1$, only the 1 term is independent. We can then explicitly work out the traces,

$$\langle \mathbf{D}, v\,|\mathbf{D}, v\rangle \;=\; \mathrm{tr}\Big(\overline{\mathbf{D}}(v)\mathbf{D}(v)[A + \overbrace{B\ \not v + \cdots}^{\substack{\text{not}\\ \text{independent}}}]\Big) \tag{2.78}$$

$$\mathrm{tr}\left(\overline{D}(v)D(v)\right) \;=\; \mathrm{tr}\left(\overline{D}(v,\varepsilon)D(v,\varepsilon)\right) = -2m_D \tag{2.79}$$

$$\mathrm{tr}\left(\overline{D}(v,\varepsilon)D(v)\right) \;=\; \mathrm{tr}\left(\overline{D}(v)D(v,\varepsilon)\right) = 0 \tag{2.80}$$

For example, in detail, we have

$$\begin{aligned} &\mathrm{tr}\left(\overline{D}(v,\varepsilon)D(v,\varepsilon)\right) \\ &= \;\mathrm{tr}\left(\sqrt{m_D}\,\varepsilon^*_\mu\gamma^\mu\,\frac{1+\not v}{2}\,\sqrt{m_D}\,\frac{1+\not v}{2}\ \not\varepsilon\right) \\ &= m_D\,\mathrm{tr}\left(\varepsilon^*_\mu\gamma^\mu\,\frac{1+\not v}{2}\ \not\varepsilon\right) \\ &= \frac{1}{2}m_D\,\mathrm{tr}\left(\varepsilon^*_\mu\gamma^\mu\ \ \not\varepsilon\right) = -2m_D \end{aligned} \tag{2.81}$$

Thus

$$\mathrm{tr}\left(\overline{\mathbf{D}}(v)\mathbf{D}(v)\right) \;=\; -2m_D\left(|d|^2 + \sum_\varepsilon |d_\varepsilon|^2\right) \tag{2.82}$$

and the result is

$$-2m_D A\left(|d|^2 + \sum_\varepsilon |d_\varepsilon|^2\right) \tag{2.83}$$

This means that (2.69) works and we got the relative normalization right.
Example 2 — next consider the matrix element of the light quark vector current,

$$\begin{aligned} \langle \mathbf{D}, v|\,\bar{q}\gamma^\mu q\,|\mathbf{D}, v\rangle \;&=\; \mathrm{tr}\Big(\overline{\mathbf{D}}(v)\mathbf{D}(v)[A\,v^\mu + \overbrace{B\,\gamma^\mu + \cdots}^{\substack{\text{not}\\ \text{independent}}}]\Big) \\ &= -2A\,m_D v^\mu\left(|d|^2 + \sum_\varepsilon |d_\varepsilon|^2\right) \end{aligned} \tag{2.84}$$

Just as in example 1, we contract the heavy quark indices. Then put into the trace everything that looks like a Lorentz vector. But all of these just reduce to v^μ because of $\mathbf{D}(v)\ \not{v} = -\mathbf{D}(v)$.

Here we know the answer!

How? What is A?

Answer

Because $\overline{q}\gamma^\mu q$ is a conserved current, its forward matrix element is determined by the charge, which here is -1 because the state contains one light antiquark

$$\begin{aligned} &\langle \mathbf{D}, v|\, \overline{q}\gamma^\mu q\, |\mathbf{D}, v\rangle \\ &= -2p\mu \left(|d|^2 + \sum_\varepsilon |d_\varepsilon|^2 \right) \\ &= -2m_D v^\mu \left(|d|^2 + \sum_\varepsilon |d_\varepsilon|^2 \right) \end{aligned} \tag{2.85}$$

$$A = 1$$

Thus the result of the spin symmetry in this case is nothing that we do not already know. However, for $\overline{q}D^\mu q$, for example, the spin symmetry argument would be exactly the same, and gives the same matrix elements up to a constant. Thus for the current $\overline{q}D^\mu q$, the relation between the D and D^* states is a nontrivial consequence of the $SU(2)^\infty \otimes \text{Lorentz}$ symmetry.

Example 3 — let's work out the matrix element of an axial vector current. Once again, there is only one invariant, which looks like this (note that $v^\mu\gamma_5$ vanishes because of $\mathbf{D}(v)\ \not{v} = -\mathbf{D}(v)$):

$$\begin{aligned} \langle \mathbf{D}, v|\, \overline{q}\gamma^\mu\gamma_5 q\, |\mathbf{D}, v\rangle &= \operatorname{tr}\Big(\overline{\mathbf{D}}(v)\mathbf{D}(v)[A\,\gamma^\mu\gamma_5]\Big) \\ &= -2A\, m_D \sum \varepsilon\, (d_\varepsilon^* d \varepsilon^{\mu *} + d^* d_\varepsilon \varepsilon^\mu) \end{aligned} \tag{2.86}$$

From this we extract.

$$\langle D^*(v,\varepsilon)|\, \overline{q}\gamma^\mu\gamma_5 q\, |D(v)\rangle = -2A\, m_D \varepsilon^{\mu *} \tag{2.87}$$

$$\langle D(v)|\, \overline{q}\gamma^\mu\gamma_5 q\, |D^*(v,\varepsilon)\rangle = -2A\, m_D \varepsilon^{\mu} \tag{2.88}$$

This result is not interesting because the answer couldn't have been anything else, but there is a moral hidden in the computation. The $\gamma^\mu\gamma_5$ appears in the trace, not just because it's in the current — it is simply the most general thing consistent with Lorentz invariance and Parity and $\mathbf{D}(v)\ \not{v} = -\mathbf{D}(v)$! In fact, the argument is precisely the same for a current like $i\,\overline{q}\,D^\mu\gamma_5\, q$.

Now for a trick question. What is

$$\langle \mathbf{D}', v'|\, \overline{q}q\, |\mathbf{D}, v\rangle \tag{2.89}$$

for $v' \neq v$?

The answer is 0! In the low energy theory, by definition, QCD interactions can't change heavy quark velocities. This is forbidden by the velocity superselection rule. Such a change in velocities would require a large, $\mathcal{O}(m_c)$, transfer of energy from the brown muck to the heavy quarks, which would take us out of the effective theory.

Now for a more serious question, indeed, related to the central question of these lectures. What is

$$\langle \mathbf{D}', v' | \, \overline{c_{v'}} \Gamma c_v \, | \mathbf{D}, v \rangle \tag{2.90}$$

where Γ is some 4×4 matrix like γ^μ or $\sigma^{\mu\nu}\gamma_5$ (only 2×2 really counts — because of $\not{v}\,\mathbf{D}(v) = \mathbf{D}(v)$ and $\overline{\mathbf{D}}(v,)\ \not{v}' = \overline{\mathbf{D}}(v')$ the space is really 2×2)?

There are two equivalent ways of thinking of thinking about this matrix element. We can figure out how $\overline{c_{v'}}\Gamma c_v$ transforms for some fixed Γ and follow the rules, or what is more convenient, we can extend the rules to incorporate the notion of **"tensor operators"**. This goes as follows. If Γ were a field transforming as

$$\Gamma \to e^{-i\vec{\epsilon}' \cdot \vec{S}^{v'}} \Gamma e^{i\vec{\epsilon} \cdot \vec{S}^v} \tag{2.91}$$

then the operator would be invariant under the spin symmetries. A field Γ transforming this way is a tensor. It is a kind of wave function for the current operator. If we imagine that we are computing the matrix element of the current multiplied by this tensor field, Γ, we can impose the spin symmetry by simply requiring invariance under all of the relevant $SU(2)^v$'s. Then at the end of the day, we can set Γ equal to whatever constant value we are interested in, and we will have the right symmetry structure.

This leads to an improved rule for computing matrix elements in low energy theory

$$\langle \mathbf{D}', v' | \, \mathcal{O} \, | \mathbf{D}, v \rangle \tag{2.92}$$

1. Replace the $|\mathbf{D}, v\rangle$ by the wave function $\mathbf{D}(v)$.
2. Replace the $\langle \mathbf{D}', v'|$ by the wave function $\overline{\mathbf{D}}'(v')$
3. Replace the operators by tensors transforming appropriately under $SU(2)^\infty$.
4. Put the wave functions together into invariants under $SU(2)^\infty$, transforming appropriately under Lorentz transformations. Still, for each independent invariant, we must include an arbitrary function of the Lorentz invariants. Each of these must be fixed by the dynamics (or some other argument).

Let's do it! The argumentation should now be familiar. We just have to contract both sets of spin indices to make an invariant both under $SU(2)^v$ and $SU(2)^{v'}$. As before there is only one term. This time, we have more objects that can contribute, because there are two vectors, v and v', but when we impose $\mathbf{D}(v)\ \not{v} = -\mathbf{D}(v)$ and

$\not{v}'\overline{\mathbf{D}}(v') = -\overline{\mathbf{D}}(v')$, they all reduce to a single unknown function of the invariant, $v'v$.

$$
\begin{aligned}
&\langle \mathbf{D}, v'|\, \overline{c_{v'}}\Gamma c_v \,|\mathbf{D}, v\rangle \\
&= \operatorname{tr}\Big(\overline{\mathbf{D}}(v')\Gamma\mathbf{D}(v)[-\xi(v'v) + \overbrace{\chi(v'v)\ \not{v} + \cdots}^{\substack{\text{not}\\ \text{independent}}}]\Big) \\
&= -\xi(v'v)\ \operatorname{tr}\Big(\overline{\mathbf{D}}(v')\Gamma\mathbf{D}(v)\Big)
\end{aligned}
\tag{2.93}
$$

If we choose the normalization right, this $\xi(v'v)$ is the Isgur-Wise function that we discussed in the first lecture. As we will show shortly, the last line of (2.93) produces the right normalization (that is why we included the minus sign, as well as calling the coefficient $\xi(v'v)$ with no extra factors).

We can get the normalization information by looking at the forward matrix element of $\Gamma = \gamma^\mu$, the vector current:

$$
\begin{aligned}
&\langle \mathbf{D}, v'|\, \overline{c_{v'}}\gamma^\mu c_v \,|\mathbf{D}, v\rangle \\
&= -\xi(v'v)\ \operatorname{tr}\Big(\overline{\mathbf{D}}(v')\gamma^\mu\mathbf{D}(v)\Big)
\end{aligned}
\tag{2.94}
$$

We must simply insert into (2.93) the expressions for $\mathbf{D}(v)$ and $\overline{\mathbf{D}}(v')$,

$$
\mathbf{D}(v) = \sqrt{m_D}\,\frac{1+\not{v}}{2}\left(-d\gamma_5 + \sum_\varepsilon d_\varepsilon\ \not{\varepsilon}\right) \tag{2.95}
$$

$$
\mathbf{D}'(v') = \sqrt{m_D}\left(d'^*\gamma_5 + \sum_{\varepsilon'} d'^*_{\varepsilon'}\varepsilon'^*_\mu\gamma^\mu\right)\frac{1+\not{v}'}{2} \tag{2.96}
$$

This gives a complicated but interesting expression that we will compute in a moment in the context of $b \to c$ transitions. But at $v' = v$ it is simple

$$
= \xi(1)\, 2m_D v^\mu\left(d'^* d - \sum_{\varepsilon',\varepsilon} d'^*_{\varepsilon'} d_\varepsilon (\varepsilon'^*_\mu\varepsilon^\mu)\right) \tag{2.97}
$$

But now we know that $\xi(1) = 1$ because

$$
\overline{c_v}\gamma^\mu c_v = v^\mu\,\overline{c}_v c_v \tag{2.98}
$$

is the symmetry current that counts the heavy c_v quarks. Thus (2.93) is correct for the conventionally normalized Igsur-Wise function that is 1 at the Shifman-Voloshin point, $v'v = 1$.

Phenomenologically more interesting are the $\overline{c}_{v'}\Gamma b_v$ currents. The analysis of their matrix elements in the effective low energy theory is entirely analogous to the discussion above. Just make the replacements $c \to b$, $D \to \overline{B}$ and $\mathbf{D} \to \mathbf{B}$. This works because in the theory with both a heavy b and a heavy c (which we are in when we are below the scale m_c) there is actually an $SU(4)^\infty \otimes$ Lorentz symmetry that behaves in exactly the same way as the $SU(2)^\infty \otimes$ Lorentz symmetry that we used above. If we wanted to, we could write all the wave functions in a 8 dimensional space, but the symmetry that rotates the b into the c is so trivial that it hardly warrants this excessively formal notation. We will simply remember that we could do it if we wanted to. The result for the transition matrix element looks exactly like (2.93)

$$\begin{aligned} &\langle \mathbf{D}', v' | \, \bar{c}_{v'} \Gamma b_v \, | \mathbf{B}, v \rangle \\ &= \xi(v'v) \sqrt{m_D m_B} \, \operatorname{tr} \left[\left(d'^* \gamma_5 + \sum_{\varepsilon'} d'^*_{\varepsilon'} \varepsilon'^*_\mu \gamma^\mu \right) \right. \\ &\left. \cdot \frac{1+ \not{v}'}{2} \Gamma \frac{1+ \not{v}}{2} \left(-b\gamma_5 + \sum_{\varepsilon} b_\varepsilon \not{\varepsilon} \right) \right] \end{aligned} \tag{2.99}$$

For example, we can extract the matrix elements for $\Gamma = \gamma^\mu$ and $\Gamma = \gamma^\mu \gamma_5$ that are relevant to the dominant semileptonic weak decays of the $\overline{B}$.

In the low energy theory, it is exactly the same Isgur-Wise function that comes into (2.99) and (2.93). This, at low energies, depends only on the brown muck. Formally, in the effective low energy theory, this is a consequence of the $SU(4)^\infty \otimes$ Lorentz symmetry.

Working out the relevant matrix elements explicitly, we find

$$\begin{aligned} &\langle D, v' | \, \bar{c}_{v'} \gamma^\mu b_v \left| \overline{B}, v \right\rangle \\ &= \xi(v'v) \sqrt{m_D m_B} \, \operatorname{tr} \left(\gamma_5 \frac{1+ \not{v}'}{2} \gamma^\mu \frac{1+ \not{v}}{2} \gamma_5 \right) \\ &= \xi(v'v) \sqrt{m_D m_B} \, \left(v^\mu + v'^\mu \right) \end{aligned} \tag{2.100}$$

$$\begin{aligned} &\langle D^*, \varepsilon', v' | \, \bar{c}_{v'} \gamma^\mu b_v \left| \overline{B}, v \right\rangle \\ &= \xi(v'v) \sqrt{m_D m_B} \, \operatorname{tr} \left(\varepsilon'^*_\nu \gamma^\nu \frac{1+ \not{v}'}{2} \gamma^\mu \frac{1+ \not{v}}{2} \gamma_5 \right) \\ &= \xi(v'v) \sqrt{m_D m_B} \, \frac{\varepsilon'^*_\nu}{4} \operatorname{tr} \left(\gamma^\nu \not{v}' \gamma^\mu \not{v} \gamma_5 \right) \\ &= i \xi(v'v) \sqrt{m_D m_B} \, \epsilon^{\mu\nu\alpha\beta} \, \varepsilon'^*_\nu v_\alpha v'_\beta \end{aligned} \tag{2.101}$$

because

$$\operatorname{tr} \left(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta \gamma_5 \right) = i \, \epsilon^{\mu\nu\alpha\beta} \tag{2.102}$$

and

$$
\begin{aligned}
&\left\langle D^*, \varepsilon', v' \right| \bar{c}_{v'} \gamma^\mu \gamma_5 b_v \left| \overline{B}, v \right\rangle \\
&= \xi(v'v)\, \sqrt{m_D m_B}\ \mathrm{tr}\left(\varepsilon'^*_\nu \gamma^\nu \frac{1+ \not{v}'}{2} \gamma^\mu \gamma_5 \frac{1+ \not{v}}{2} \gamma_5 \right) \\
&= \xi(v'v)\, \sqrt{m_D m_B}\ \mathrm{tr}\left(\varepsilon'^*_\nu \gamma^\nu \frac{1+ \not{v}'}{2} \gamma^\mu \frac{1- \not{v}}{2} \right) \\
&= \xi(v'v)\, \sqrt{m_D m_B}\ \left[(1+v'v)\, \varepsilon'^{\mu *} - \varepsilon'^{\nu *} v_\nu v'^\mu \right]
\end{aligned}
\tag{2.103}
$$

Matching and Running

My object all sublime
I shall achieve in time —
To make the punishment fit the crime —
The punishment fit the crime

The Mikado
Gilbert and Sullivan

So far, we have been at low energies — $k^\mu, \mu \ll m_b, m_c$. To complete the calculation of matrix elements between heavy meson states, we must answer the following question. How are operators in the low energy theory related to currents in the full theory? In particular, consider the relation between $\bar{c}\Gamma b$ and $\bar{c}_{v'}\Gamma b_v$.

The effective field theory formalism gives us a crank to turn to find this relation, shown in the figure below.

We begin at high energies, where we know the form of the semileptonic weak interactions in terms of the currents in the full QCD theory. We start by matching the physics in two theories on either side of the scale $\mu = m_b$

High energies	$\downarrow$
normal quarks	
$\mu = m_b$	match physics
b heavy	$\downarrow$
	RG running
c normal	$\downarrow$
$\mu = m_c$	match physics
both heavy	$\downarrow$
$\mu \approx \Lambda$	compute m.e.

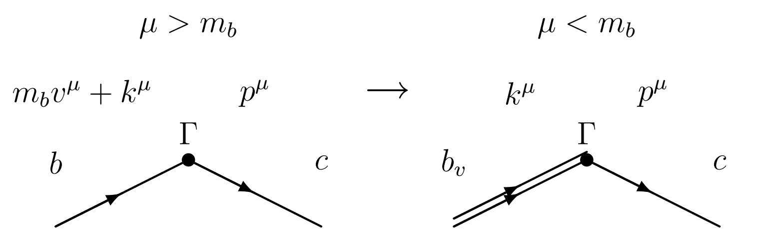
There are two approximations involved — $1/m_b$ and $\alpha_s(m_b)$.

In leading order in both α_s and $1/m_b$, the matching is trivial:

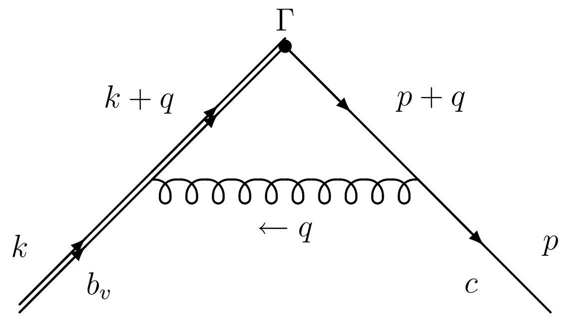
$$b(x) \to e^{-im_b\, v_\mu x^\mu} b_v(x) \tag{3.104}$$

$$\bar{c}\Gamma b \to e^{-im_b\, v_\mu x^\mu}\, \bar{c}\Gamma b_v$$

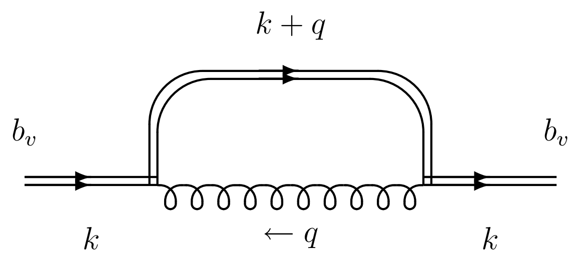
In momentum space, the $e^{-im_b\, v_\mu x^\mu}$ simply eliminates all effects of order m_b, as shown diagrammatically below:



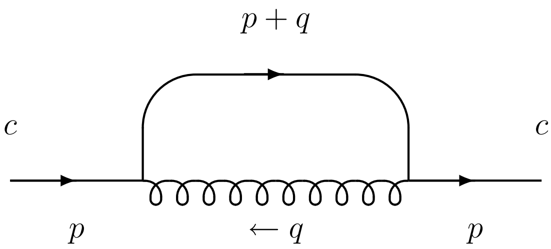
For $\mu > m_b$, $\bar{c}\Gamma b$ doesn't depend on μ (because it is the Noether current associated with a softly broken symmetry), but in low energy theory, $m_c < \mu < m_b$, there is no symmetry. Thus we expect that the current has a nonzero anomalous dimension. To compute it, we need three Feynman graphs



and the self energies of the b_v and c



and



We will compute them in reverse order, using dimensional regularization (DR) and minimal subtraction (we'll only worry about anomalous dimensions to one loop, so there is no difference between MS and $\overline{MS}$).

The tools we need will be the standard ones. Combining denominators, with the Feynman trick, and extending the theory to $n = 4 - \epsilon$ dimensions, for regularization, where the typical Feynman integral looks like

$$\begin{aligned} &\int \frac{d^n\ell}{(2\pi)^n} \frac{(\ell^2)^\beta}{(\ell^2 - A^2)^\alpha} \\ &= \frac{i}{(4\pi)^{n/2}} (-1)^{\alpha+\beta} (A^2)^{\beta-\alpha+n/2} \\ &\cdot \frac{\Gamma(\beta + n/2)\,\Gamma(\alpha - \beta - n/2)}{\Gamma(n/2)\Gamma(\alpha)} \end{aligned} \tag{3.105}$$

The issue of infrared (IR) divergences in dimensional regularization is sometimes confusing, so I will begin with a digression.

The physical idea of a regularization scheme is that it is a modification of the physics of the theory at short distances that allows us to calculate the quantum corrections. If we modify the physics only at short distances, we expect that all the effects of the regularization can be absorbed into the parameters of the theory. That is how we chose the parameters in the first place. However, it is not obvious that DR is a modification of the physics at short distances. To see to what extent it is, consider a typical Feynman graph in the unregularized theory in Euclidean space. In one loop (which I discuss for simplicity), all graphs ultimately reduce to sums of objects of the following form:

$$I = \int [dx] \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 + A^2)^\alpha}\,, \tag{3.106}$$

α is some integer

In DR, these get replaced by integrals over $4 + \delta$ dimensional momentum space. I am going to think of $\epsilon = -\delta$ as being negative. This doesn't really matter, because everything is defined by analytic continuation anyway, but it makes things easier to talk about. The regularized integrals have the form

$$I_\delta = c(\delta) \int [dx] \, \frac{d^{4+\delta}\ell}{\mu^\delta \, (2\pi)^{4+\delta}} \, \frac{1}{(\ell_\delta^2 + \ell^2 + A^2)^\alpha} \,, \tag{3.107}$$

where $c(\delta) \to 1$ as $\delta \to 0$, and where I have explicitly separated out the "extra" δ dimensions, so that ℓ^2 is the 4 dimensional length.

In practice, we would do the whole n dimensional integral at once, using (3.105). However, to see what is happening, I am going to split the integral up into an integral over the usual 4 dimensions and the extra δ dimensions. Rewrite the integral as follows:

$$c(\delta) \int [dx] \, \frac{d^\delta \ell}{(2\pi\mu)^\delta} \, \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{(\ell_\delta^2 + \ell^2 + A^2)^\alpha} \,, \tag{3.108}$$

Now do integral over δ extra dimensions

$$I_\delta = \int [dx] \, \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{(\ell^2 + A^2)^\alpha} \, r(\delta) \left(\frac{\ell^2 + A^2}{4\pi\mu^2} \right)^{\delta/2} \,, \tag{3.109}$$

where

$$r(\delta) = c(\delta) \, \frac{\Gamma(\alpha - \delta/2)}{\Gamma(\alpha)} \,. \tag{3.110}$$

The factor $r(\delta) \to 1$ as $\delta \to 0$. The important factor is

$$\rho^{\delta/2} \quad \text{where} \quad \rho = \frac{\ell^2 + A^2}{4\pi\mu^2} \,. \tag{3.111}$$

This also $\to 1$ as $\delta \to 0$, but here convergence depends on ℓ and A.

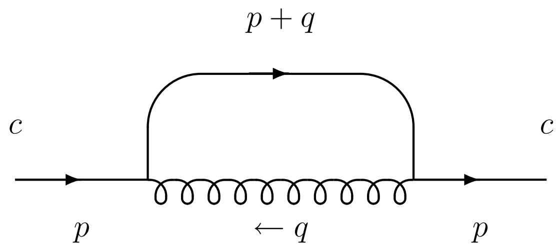
$$\rho^{\delta/2} = e^{(\delta \ln \rho)/2} \,, \tag{3.112}$$

$$\rho^{\delta/2} \approx 1 \quad \text{for} \quad |\ln \rho| \ll \frac{1}{\delta} \,. \tag{3.113}$$

You can see from (3.113), that for very small δ, we have not changed the physics for ℓ (the loop momentum) and A (which involves external momenta and masses) of the order of μ, but that there are significant differences if either ℓ or A is much larger than μ for fixed δ, **or if they are both much smaller than μ.** The first is exactly what we want. This is just a modification of the physics at short distances. The second is the problem. DR can modify the physics at large distances as well, so that, in general, it is not a sensible regulator.

However, we are OK so long as we avoid IR divergences. Then small momenta don't contribute in the integrals (A cannot get very small). We can always do this in the calculation of an anomalous dimension or a matching contribution. In matching, it is trivial. The matching is chosen so that the infrared physics is exactly the same in the two effective theories, so that the IR divergences always cancel in the calculation. In calculating an anomalous dimension, the IR region is also irrelevant. For example, we can keep external momenta nonzero until after renormalization, eliminating IR divergences. However, we will do this in out heads, because the final result for the anomalous dimension doesn't depend on the external momentum.

First consider the light quark self energy:



In Feynman gauge and $n = 4 - \epsilon$ dimensions, the corresponding Feynman integral is

$$\int \frac{d^n q}{(2\pi)^n} \frac{-i}{q^2 + i\epsilon} \left[(-ig\mu^{\epsilon/2}) T^a \gamma^\nu \frac{i(\not{q} + \not{p})}{(q+p)^2 + i\epsilon} (-ig\mu^{\epsilon/2}) T^a \gamma_\nu \right] \tag{3.114}$$

$$= -\frac{4}{3} g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{\gamma^\nu (\not{q} + \not{p}) \gamma_\nu}{(q+p)^2 q^2} \tag{3.115}$$

I've dropped $i\epsilon$

We are only interested in the $1/\epsilon$ and $\ln \mu$ terms, so we can do Dirac manipulations for $\epsilon = 0$, to get

$$= \frac{8}{3} g^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{(\not{q} + \not{p})}{(q+p)^2 q^2} \tag{3.116}$$

Then combining denominators using the Feynman trick gives

$$\frac{1}{(q+p)^2 q^2} = \int_0^1 dx \, \frac{1}{\left[q^2 + 2xpq + xp^2\right]^2} \tag{3.117}$$

and $q \to \ell - xp$

$$= \frac{8}{3} g^2 \mu^\epsilon \int \frac{d^n \ell}{(2\pi)^n} \int_0^1 dx \, \frac{(\not{\ell} + (1-x)\not{p})}{\left[\ell^2 + x(1-x)p^2\right]^2} \tag{3.118}$$

$$= \frac{4}{3} g^2 \mu^\epsilon \;\; \not{p} \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{\left[\ell^2\right]^2} \tag{3.119}$$

where I finally dropped the external momentum dependence, which was interesting only because it was acting as an IR cutoff. The using (3.107), and $\Gamma(\epsilon) = \frac{1}{\epsilon} + \cdots$ we get the final result for the $1/\epsilon$ term,

$$\frac{8}{3}\frac{g^2}{16\pi^2}\frac{\mu^\epsilon}{\epsilon} i \not{p} \tag{3.120}$$

Then after the $1/\epsilon$ pole is removed by MS, the two point function is

$$\Gamma_2^c = i \not{p} \left(1 + \frac{8}{3}\frac{g^2}{16\pi^2} \ln \mu + \cdots\right) \tag{3.121}$$

From this we can calculate the anomalous dimension by requiring that RGE be satisfied

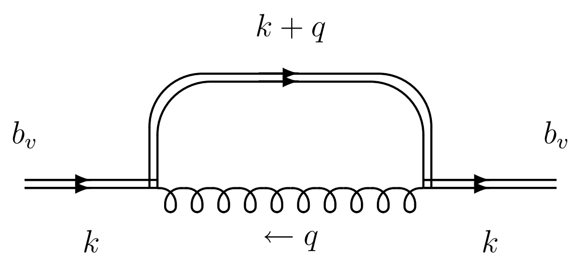
$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_c\right) \Gamma_2^c = 0 \tag{3.122}$$

The β function is $\beta(g) = \mathcal{O}(g^3)$ and so irrelevant, thus

$$\gamma_c = -\frac{4}{3}\frac{g^2}{16\pi^2} \tag{3.123}$$

This should be familiar.

Now let's do the same thing for self energy of the b_v:



$$\begin{aligned} &\int \frac{d^nq}{(2\pi)^n}\frac{-i}{q^2+i\epsilon} \\ &\left[(-ig\mu^{\epsilon/2})T^a v^\nu \frac{i}{qv+kv+i\epsilon}(-ig\mu^{\epsilon/2})T^a v_\nu\right] \\ &= -\frac{4}{3}g^2 \int \frac{d^nq}{(2\pi)^n}\frac{1}{(q^2)(qv+kv)} \end{aligned} \tag{3.124}$$

I expect a result proportional to (kv), the tree term in the 2-point function. To isolate it, expand in (kv), keep the linear term, then set $k = 0$ (except for its job as an IR regulator):

$$= \frac{4}{3}g^2\mu^\epsilon\,(kv) \int \frac{d^nq}{(2\pi)^n}\frac{1}{(q^2)(qv)^2} \tag{3.125}$$

There are lots of ways of doing these funny looking Feynman integrals. I am going to show you a method that, I think, always works, at least for finding anomalous dimensions. After you have combined the normal Feynman propagators, combine them with the funny looking denominators using the following identity:

$$\frac{1}{(q^2)^n (qv)^m} = \frac{(n+m-1)!}{(n-1)!\,(m-1)!} \int_0^\infty \frac{2^m \lambda^{m-1} d\lambda}{(q^2 + 2\lambda qv)^{n+m}} \tag{3.126}$$

Here, this gives

$$= \frac{32}{3} g^2 \mu^\epsilon \,(kv) \int \frac{d^n q}{(2\pi)^n} \int_0^\infty \frac{\lambda\, d\lambda}{(q^2 + 2\lambda qv)^3} \tag{3.127}$$

Next shift to eliminate linear terms in q, $q = \ell - \lambda v$

$$= \frac{16}{3} g^2 \mu^\epsilon \,(kv) \int \frac{d^n \ell}{(2\pi)^n} \int_0^\infty \frac{d\lambda^2}{(\ell^2 - \lambda^2)^3} \tag{3.128}$$

Now rescale to a dimensionless integration variable, κ, pulling out a factor of $\sqrt{-\ell^2}$ ($\ell^2 < 0$ because we are in Euclidean space in our Feynman integrals)

$$\lambda^2 = -\ell^2 \,\kappa^2 \tag{3.129}$$

which gives

$$= -\frac{16}{3} g^2 \mu^\epsilon \,(kv) \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{(\ell^2)^2} \cdot \int_0^\infty \frac{d\kappa^2}{(1+\kappa^2)^3} \tag{3.130}$$

This breaks the integral up into two pieces, one of which is the normal log divergent (in $n = 4$) part, and the other of which is a convergent integral over κ.

Doing the κ integral is trivial, and gives

$$= -\frac{8}{3} g^2 \mu^\epsilon \,(kv) \int \frac{d^n \ell}{(2\pi)^n} \frac{1}{(\ell^2)^2} \tag{3.131}$$

$$= -\frac{16}{3} \frac{g^2}{16\pi^2} \frac{\mu^\epsilon}{\epsilon} i(kv) + \cdots \tag{3.132}$$

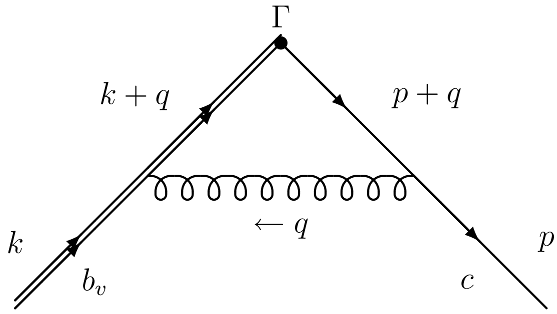
Then after renormalization, the two point function is

$$\Gamma_2^{b_v} = i(kv) \left(1 - \frac{16}{3} \frac{g^2}{16\pi^2} \ln \mu + \cdots \right) \tag{3.133}$$

and using the RGE as before leads to

$$\gamma_{b_v} = \frac{8}{3} \frac{g^2}{16\pi^2} \tag{3.134}$$

Finally. look at the three point function.



Setting external momenta to zero, the Feynman integral is

$$
\begin{aligned}
&\int \frac{d^n q}{(2\pi)^n} \frac{-i}{q^2} \\
&\left[(-ig\mu^{\epsilon/2})T^a\gamma^\nu \frac{i\,\not{q}}{q^2}\Gamma\frac{i}{qv}(-ig\mu^{\epsilon/2})T^a v_\nu\right] \\
&= -i\frac{4}{3}g^2\mu^\epsilon \;\not{v} \int \frac{d^n q}{(2\pi)^n}\frac{\not{q}}{(q^2)^2(qv)}\,\Gamma \\
&= -i\frac{16}{3}g^2\mu^\epsilon \;\not{v} \int \frac{d^n q}{(2\pi)^n}\int_0^\infty d\lambda \frac{\not{q}}{(q^2+2\lambda qv)^3}\,\Gamma \\
&= -i\frac{16}{3}g^2\mu^\epsilon \;\not{v} \int \frac{d^n \ell}{(2\pi)^n}\int_0^\infty d\lambda \frac{\not{\ell}-\lambda\not{v}}{(\ell^2-\lambda^2)^3}\,\Gamma \\
&= i\frac{8}{3}g^2\mu^\epsilon\,\Gamma \int \frac{d^n \ell}{(2\pi)^n}\int_0^\infty \frac{d\lambda^2}{(\ell^2-\lambda^2)^3}
\end{aligned}
\tag{3.135}
$$

This is the same integral as before, so the result for the vertex is

$$\Gamma_{\bar{c}\Gamma b_v} = \left(1 + \frac{8}{3}\,\frac{g^2}{16\pi^2}\ln\mu + \cdots\right)\Gamma \tag{3.136}$$

This must satisfy the RGE

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_c + \gamma_{b_v} - \gamma_{\bar{c}\Gamma b_v}\right)\Gamma_{\bar{c}\Gamma b_v} = 0 \tag{3.137}$$

or

$$\gamma_{\bar{c}\Gamma b_v} = 4\,\frac{g^2}{16\pi^2} \tag{3.138}$$

The β-function is

$$\mu\frac{\partial}{\partial\mu}g \equiv \beta(g) = -\frac{b\,g^3}{16\pi^2} = -\frac{33-2n_q}{3}\,\frac{g^3}{16\pi^2} = -\frac{25}{3}\,\frac{g^3}{16\pi^2} \tag{3.139}$$

where n_q the number of **light** quarks, b_v doesn't count because we cannot produce quark antiquark pairs in the effective low energy theory. If you explicitly compute the contribution of a b_v loop to the gluon self energy, you get zero, as you expect. In the rest frame, in position space, this is particularly trivial — the heavy b_0 quark can propagate only forward in time, so there is no way to make a loop.

The running coupling is

$$\frac{\alpha_s(\mu)}{4\pi} \approx \frac{1}{2b\ln\mu/\Lambda} \tag{3.140}$$

Now the matrix element $M(\mu) = \langle \mathbf{D}|\, \bar{c}\Gamma b_v \,|\mathbf{B}\rangle$ depends on μ.

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_{\bar{c}\Gamma b_v}\right) M(\mu) = 0 \tag{3.141}$$

The solution is

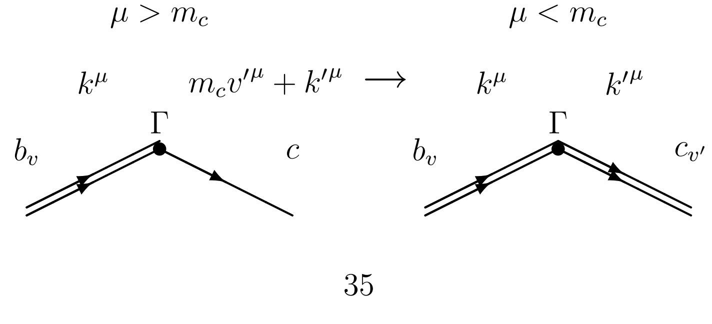
$$\begin{aligned} M(\mu) &= M(\mu_0)\cdot\exp\left(\int_{\mu_0}^{\mu}\gamma_{\bar{c}\Gamma b_v}(g(\mu'))\frac{d\mu'}{\mu'}\right) \\ &= M(\mu_0)\cdot\exp\left(\int_{g(\mu_0)}^{g(\mu)}\frac{\gamma_{\bar{c}\Gamma b_v}(g')}{\beta(g')}dg'\right) \\ &\approx M(\mu_0)\cdot\exp\left(-\frac{12}{25}(\ln g(\mu) - \ln g(\mu_0))\right) \\ &= M(\mu_0)\cdot\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{6}{25}} \end{aligned} \tag{3.142}$$

$M(m_b)$ is related to the matrix element in the full theory

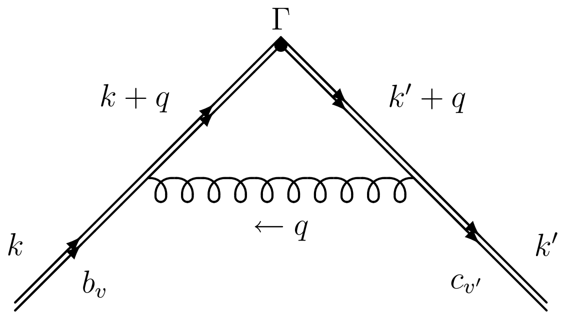
$$M(m_b) = M(\mu)\cdot\left(\frac{\alpha_s(m_b)}{\alpha_s(\mu)}\right)^{-\frac{6}{25}} \tag{3.143}$$

Thus we pick up a multiplicative factor in running down from m_b. We are on our way to low energies, so we run all the way down to m_c, then switch to the next effective theory, in which the c quark is treated as heavy, $c \to c_v$. Again, we match the physics in the two theories at the boundary, at scale $\mu = m_c$, and break up the c quark momentum in the full theory into $p = m_c v' + k'$. In leading order, the matching looks like

$$\begin{aligned} c(x) &\to e^{-im_c v'_\mu x^\mu} c_{v'}(x) \\ \bar{c}\Gamma b_v &\to e^{im_c v'{}_\mu x^\mu}\,\overline{c_{v'}}\Gamma b_v \end{aligned} \tag{3.144}$$



Now finally, we must do the most interesting anomalous dimension calculation, for the anomalous dimension of $\overline{c_{v'}}\Gamma b_v$.[17]



Again setting external momenta to zero, the integral is

$$\begin{aligned}\int \frac{d^n q}{(2\pi)^n}\frac{-i}{q^2}\left[(-ig\mu^{\epsilon/2})T^a v'^\nu \frac{i}{qv'}\Gamma\frac{i}{qv}(-ig\mu^{\epsilon/2})T^a v_\nu\right] \\ = -i\frac{4}{3}g^2\mu^\epsilon\, v'v \int \frac{d^n q}{(2\pi)^n}\frac{\Gamma}{(q^2)(qv)(qv')} \\ = -i\frac{32}{3}g^2\mu^\epsilon\, v'v \cdot \int \frac{d^n q}{(2\pi)^n}\int_0^\infty d\lambda \int_0^\infty d\lambda'\, \frac{\Gamma}{(q^2+2\lambda qv+2\lambda' qv')^3}\end{aligned} \tag{3.145}$$

$$= -i\frac{32}{3}g^2\mu^\epsilon\, v'v \cdot \int \frac{d^n \ell}{(2\pi)^n}\int_0^\infty d\lambda \int_0^\infty d\lambda'\, \frac{\Gamma}{\left[\ell^2-(\lambda^2+\lambda'^2+2v'v\lambda\lambda')\right]^3} \tag{3.146}$$

Now rescale

$$\lambda = \sqrt{-\ell^2}\,\kappa\,, \quad \lambda' = \sqrt{-\ell^2}\,\kappa' \tag{3.147}$$

giving

$$= i\frac{32}{3}g^2\mu^\epsilon\, v'v \int \frac{d^n \ell}{(2\pi)^n}\frac{1}{(\ell^2)^2}\cdot\int_0^\infty d\kappa \int_0^\infty d\kappa'\, \frac{\Gamma}{\left(1+\kappa^2+\kappa'^2+2v'v\kappa\kappa'\right)^3} \tag{3.148}$$

Now we can do the $d^n\ell$ integral

$$\begin{aligned}&= -\frac{64}{3}\,\frac{g^2}{16\pi^2}\,\frac{\mu^\epsilon}{\epsilon}\,v'v\cdot\int_0^\infty d\kappa\int_0^\infty d\kappa'\,\frac{\Gamma}{\left(1+\kappa^2+\kappa'^2+2v'v\kappa\kappa'\right)^3} \\ &= -\frac{32}{3}\,\frac{g^2}{16\pi^2}\,\frac{\mu^\epsilon}{\epsilon}\,v'v\cdot\int_0^\infty d\kappa^2\int_0^{\pi/2} d\theta\,\frac{\Gamma}{[1+\kappa^2(1+v'v\sin 2\theta)]^3} \\ &= -\frac{16}{3}\,\frac{g^2}{16\pi^2}\,\frac{\mu^\epsilon}{\epsilon}\,v'v\,\Gamma\cdot\int_0^{\pi/2}\frac{d\theta}{(1+v'v\sin 2\theta)} \\ &= -\frac{16}{3}\,\frac{g^2}{16\pi^2}\,\frac{\mu^\epsilon}{\epsilon}\,v'v\,r(v'v)\,\Gamma\end{aligned} \tag{3.149}$$

where

$$r(w) = \frac{\ln\left(w + \sqrt{w^2-1}\right)}{\sqrt{w^2-1}} \tag{3.150}$$

The θ integral is best done by a symbolic manipulation program, however, if you hunker down to it, it is not so hard.

$$\int_0^{\pi/2} \frac{d\theta}{(1 + w\sin 2\theta)} \tag{3.151}$$

let $z = e^{2i\theta}$, integral can be deformed to real axis

$$\begin{aligned}
&= \int_1^{-1} \frac{dz}{2iz} \frac{1}{\left(1 + \frac{w}{2i}[z - z^{-1}]\right)} \\
&= \int_1^{-1} \frac{dz}{w} \frac{1}{(z^2 + 2iz/w - 1)} \\
&= \int_1^{-1} \frac{dz}{w} \frac{1}{\left(z + \frac{i}{w} - \frac{\sqrt{w^2-1}}{w}\right)\left(z + \frac{i}{w} + \frac{\sqrt{w^2-1}}{w}\right)} \\
&= \frac{1}{2\sqrt{w^2-1}} \int_1^{-1} \left[\frac{1}{\left(z + \frac{i}{w} - \frac{\sqrt{w^2-1}}{w}\right)} - \frac{1}{\left(z + \frac{i}{w} + \frac{\sqrt{w^2-1}}{w}\right)}\right] \\
&= \frac{1}{2\sqrt{w^2-1}} \ln\left(\frac{\left(-1 + \frac{i}{w} - \frac{\sqrt{w^2-1}}{w}\right)}{\left(1 + \frac{i}{w} - \frac{\sqrt{w^2-1}}{w}\right)} \frac{\left(1 + \frac{i}{w} + \frac{\sqrt{w^2-1}}{w}\right)}{\left(-1 + \frac{i}{w} + \frac{\sqrt{w^2-1}}{w}\right)}\right)
\end{aligned} \tag{3.152}$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{w^2-1}} \ln\left(\frac{(w + \sqrt{w^2-1})^2 + 1}{(w - \sqrt{w^2-1})^2 + 1}\right) \\
&= \frac{1}{2\sqrt{w^2-1}} \ln\left(\frac{w + \sqrt{w^2-1}}{w - \sqrt{w^2-1}}\right) \\
&= \frac{1}{\sqrt{w^2-1}} \ln\left(w + \sqrt{w^2-1}\right)
\end{aligned} \tag{3.153}$$

Thus finally, the 3-point function is

$$\Gamma_{\overline{c_{v'}}\Gamma b_v} = \left(1 - \frac{16}{3}\frac{g^2}{16\pi^2}\left(v'v\, r(v'v)\right)\ln\mu + \cdots\right)\Gamma \tag{3.154}$$

This must satisfy the RGE

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_{c'_v} + \gamma_{b_v} - \gamma_{\overline{c_{v'}}\Gamma b_v}\right)\Gamma_{\overline{c_{v'}}\Gamma b_v} = 0 \tag{3.155}$$

or

$$\gamma_{\overline{c_{v'}}\Gamma b_v} = -\frac{16}{3}\frac{g^2}{16\pi^2}\left(v'v\, r(v'v) - 1\right) \tag{3.156}$$

Note that anomalous dimension vanishes for $v'v = 1$, symmetry current, which we know on general grounds does not run.

Now solving the RGE and putting this together with the previous calculation, we have the final result for the relation between the current in the full theory and the low energy effective theory in leading order in $1/m$ and α_s

$$\overline{q}'\Gamma q = C_{q'q}(\mu)\,\overline{q}'_{v'}\Gamma q_v \tag{3.157}$$

where for $m_{q'} < m_q$

$$C_{q'q}(\mu) = \left(\frac{\alpha_s(m_q)}{\alpha_s(m_{q'})}\right)^{-\frac{6}{33-2n_q}} \left(\frac{\alpha_s(m_{q'})}{\alpha_s(\mu)}\right)^{\frac{8(v'v\,r(v'v)-1)}{33-2n_q}} \tag{3.158}$$

While we cannot compute the low energy matrix element, we expect it to given approximately by dimensional analysis if we take $\mu \approx \Lambda_{\mathrm{QCD}}$, so that the important multiplicative factor is

$$C_{q'q} = C_{q'q}(\Lambda) \tag{3.159}$$

This gives the final result:

$$\begin{aligned}&\left\langle D, v'\right|\overline{c}\gamma^\mu \mu b\left|\overline{B}, v\right\rangle\\ &= C_{cb}\,\xi(v'v)\,\sqrt{m_D m_B}\,\left(v^\mu + v'^\mu\right)\end{aligned} \tag{3.160}$$

$$\begin{aligned}&\left\langle D^*, \varepsilon', v'\right|\overline{c}\gamma^\mu b\left|\overline{B}, v\right\rangle\\ &= iC_{cb}\,\xi(v'v)\,\sqrt{m_D m_B}\,\epsilon^{\mu\nu\alpha\beta}\,\varepsilon'^*_\nu v_\alpha v'_\beta\end{aligned} \tag{3.161}$$

$$\begin{aligned}&\left\langle D^*, \varepsilon', v'\right|\overline{c}\gamma^\mu\gamma_5 b\left|\overline{B}, v\right\rangle\\ &= C_{cb}\,\xi(v'v)\,\sqrt{m_D m_B}\,\left[(1+v'v)\,\varepsilon'^{\mu *} - \varepsilon'^{\nu *} v_\nu v'^\mu\right]\end{aligned} \tag{3.162}$$

$$\begin{aligned}&\left\langle D, v'\right|\overline{c}\gamma^\mu \mu b\left|\overline{D}, v\right\rangle\\ &= C_{cc}\,\xi(v'v)\,m_D\,\left(v^\mu + v'^\mu\right)\end{aligned} \tag{3.163}$$

$$\begin{aligned}&\left\langle D^*, \varepsilon', v'\right|\overline{c}\gamma^\mu b\left|\overline{D}, v\right\rangle\\ &= iC_{cc}\,\xi(v'v)\,m_D\,\epsilon^{\mu\nu\alpha\beta}\,\varepsilon'^*_\nu v_\alpha v'_\beta\end{aligned} \tag{3.164}$$

$$\begin{aligned}&\left\langle D^*, \varepsilon', v'\right|\overline{c}\gamma^\mu\gamma_5 b\left|\overline{D}, v\right\rangle\\ &= C_{cc}\,\xi(v'v)\,m_D\,\left[(1+v'v)\,\varepsilon'^{\mu *} - \varepsilon'^{\nu *} v_\nu v'^\mu\right]\end{aligned} \tag{3.165}$$

This is the fundamental result of the heavy quark effective field theory. You can now try to improve on this by incorporating higher order effects in α_s and $1/m$. The α_s corrections are straightforward applications of QCD perturbation theory in the two theories. The $1/m$ terms, on the other hand, involve new, higher dimension operators, who matrix elements are new nonperturbative functions. This generally introduces much additional uncertainty into the game, except at the Shifman-Voloshin point [18, 19], or the spin 1/2. Λ_c and Λ_b baryons [20], where things are still relatively simple.

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